

Matrix Models and 2D String Theory

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Abstract

String theory in two-dimensional spacetime illuminates two main threads of recent development in string theory: (1) Open/closed string duality, and (2) Tachyon condensation. In two dimensions, many aspects of these phenomena can be explored in a setting where exact calculations can be performed. These lectures review the basic aspects of this system.

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1 Introduction

One of the most remarkable developments in string theory in recent years is the idea that, in certain circumstances (superselection sectors), it has a presentation as a large N gauge dynamics – gravitation is a collective phenomenon of the gauge theory, and closed strings are represented by loop observables of the gauge theory. The gauge theory in these situations provides an ansatz for the nonperturbative *definition* of the theory in that superselection sector.

By superselection sector, one means a choice of asymptotic behavior for the low-energy fields. A canonical example is string theory in $AdS_5 \times S^5$ (for a review, see [1]), where the states of the theory all have a metric that asymptotes to the anti-de Sitter metric times a round sphere, and the self-dual five-form field strength of type IIB supergravity carries N units of flux through the S^5 . The gauge theory equivalent is maximally supersymmetric $U(N)$ Yang-Mills theory in $D = 4$ spacetime dimensions. The correspondence equates states of geometry and matter in this superselection sector with states of the gauge theory. Both $AdS_5 \times S^5$ and maximally supersymmetric gauge theory possess the same global superconformal symmetry ($SU(2, 2|4)$ in the language of supergroups), which then organizes the state space into representations of the superconformal algebra. For instance, one can match the one-particle states, and the operators that create them from the vacuum, by their representation properties. The operators that create and destroy strings are represented in the gauge theory description by Wilson loops, $\text{tr}[\exp i\oint A]$, and their supersymmetric generalizations.

Upon the injection of a little energy, in gravity the generic state is a gas of supergravitons (the graviton and particles related to it by supersymmetry); if we put in a lot of energy, we expect a black hole to form. On the gauge theory side, at low energies the excitations are built from collections of gauge singlet operators (multiple ‘single-particle creation operators’) acting on the vacuum; at high energies, the gauge theory undergoes a “deconfinement transition” where energy is equipartitioned into all N^2 fields of the matrix field theory. The correspondence equates the transition from supergraviton gas to black hole on the geometry side, and the deconfinement transition on the gauge theory side [2].

Indeed, this equivalence first arose via the study of black holes carrying D-brane charge in string theory (for a review, see [3]). On the one hand, the dynamics on the branes is described at low energies by the lightest strings attached to the branes. The spacetime effective field theory, in which these strings are the quanta, is a Yang-Mills gauge theory with various matter fields. On the other hand, the branes source a geometry in which there is an increasing redshift of physics near the branes, as seen by asymptotic observers. Thus low energy also means gravitational physics near the branes. The gauge/gravity equivalence is the statement that these two descriptions have an overlapping region of validity, namely that of objects near the branes at low energies. In particular, geometrical excitations of the brane typically lead to horizon formation (‘black’ branes), whose thermodynamic properties (*c.f.*

[3, 1]) can be compared to those of the gauge theory in the cases where they can be computed.

The loop variables describing strings in the gauge theory representation are often cumbersome to work with, and it remains a problem to dig out quasi-local gravitational and other closed string physics from this exact formulation. For instance, the local physics of the horizon and singularity of black holes and black branes are not well-understood in the gauge theory language (although there is some recent progress [4]). It would be useful to have a well-developed dictionary translating between gauge theory quantities and the standard perturbative formulation of string theory as a sum over surfaces. Generally, we don't know how to read off local physics beyond qualitative statements which are dictated by symmetries (in particular, by scaling arguments) [5, 6, 7].

Part of the reason that this dictionary is poorly developed is that the correspondence is a strong/weak coupling duality. The radius of curvature R of both AdS_5 and S^5 , relative to the Planck scale ℓ_{pl} of quantum gravity, is $N = (R/\ell_{\text{pl}})^4$; relative to the scale ℓ_s set by the string tension, it is $g_{\text{YM}}^2 N = (R/\ell_s)^4$. Thus for the spacetime to have a conventional interpretation as a geometry well-approximated by classical Einstein gravity, we should work in the gauge theory at both large N and large effective ('t Hooft) coupling strength $g_{\text{YM}}^2 N$. Thus when stringy and quantum gravity fluctuations are suppressed, the gauge theory description is strongly coupled; and when the gauge theory is perturbative, the geometry has unsuppressed quantum and stringy effects.

Often in physics, useful information can be gathered by consideration of low-dimensional model systems, which hopefully retain essential features of dynamics, while simplified kinematics renders precise analysis possible. If one or another side of the duality is exactly solvable, then we can bypass the difficulty of strong/weak duality.

String theory in two spacetime dimensions provides just such an example of the gauge/gravity (or rather open string/closed string) correspondence, in which the gauge theory is an exactly solvable random matrix model, and the worldsheet description of string theory involves a conformal field theory (CFT) which has been solved by conformal bootstrap techniques.

The random matrix formulation of 2D string theory was discovered well before the recent developments involving D-branes; in fact it provided some of the motivation for the discovery of D-branes. The initial work on the matrix model is reviewed extensively in [8, 9]. The exact solution of Liouville theory was not developed at that time, and so precise comparison with worldsheet computations was rather limited in scope. The development of the conformal bootstrap for Liouville [10, 11, 12, 13, 14, 15], reviewed in [16, 17], took place in the following decade, while much of string theory research was focussed on gauge/gravity equivalence. It has only been in the last year or so that these various threads of research have been woven together [18, 19, 20, 21, 22].

Our goal in these lectures will be to provide a self-contained overview this system,

giving an introduction to the matrix model of 2D string theory, as well as the CFT techniques used to calculate the corresponding perturbative string amplitudes. We will then illustrate the map between these two presentations of 2D string theory.

Along the way, we will encounter a second major theme in recent string research – the subject of *tachyon condensation* (for reviews, see [23, 24]. A tachyon is simply terminology for an instability, a perturbation which grows exponentially instead of undergoing small oscillations. Loosely speaking, in the ‘effective potential’ of string theory, one has chosen to start the world at a local maximum of some component of the ‘string field’. By condensing this mode, one learns about the topography and topology of this effective potential, and thus about the vacuum structure of string theory.

Much effort has gone into understanding the tachyons associated to the decay of unstable collections of D-branes in string theory. Here the unstable mode or modes are (open) strings attached to the brane or branes. For example, when one has a brane and an anti-brane, the initial stages of their mutual annihilation is described by the condensation of the lightest (in this case, tachyonic) open string stretching between the brane and the anti-brane. Eventually the brane decays completely into (closed) string radiation. One might wonder whether there is a region of overlapping validity of the two descriptions, just as in the gauge/gravity (open string/closed string) correspondences described above. We will see evidence that this is the case in 2D string theory. The random matrix presentation of 2D string theory was first introduced as an alternative way to describe the worldsheets of closed 2D strings, yet the evidence suggests that it is in fact a description of the open string tachyon condensate on unstable D-particles.

The lectures are aimed at a broad audience; along the way, many ideas familiar to the practicing string theorist are summarized in order to make the presentation as self-contained as possible. We begin with a brief overview of perturbative string theory as a way of introducing our primary subject, which is string theory in two-dimensional backgrounds.

2 An overview of string theory

String theory is a generalization of particle dynamics.² The sum over random paths gives a representation of the particle propagator

$$\begin{aligned}
 G(x, x') &= \langle x' | \frac{i}{\partial^2 - m^2} | x \rangle \\
 &= \int_0^\infty dT \langle x' | e^{iT(\partial^2 - m^2)} | x \rangle \\
 &= \int_{\substack{X(0)=x \\ X(1)=x'}} \frac{\mathcal{D}g \mathcal{D}X}{\text{Diff}} \exp \left[i \int_0^1 dt \sqrt{g} [g^{tt} \partial_t X^2 + m^2] \right]. \quad (1)
 \end{aligned}$$

²For a more detailed introduction, the reader may consult the texts [25, 26].

In the second line, the use of the proper time (*Schwinger*) parametrization turns the evaluation of the propagator into a quantum mechanics problem, which can be recast as a path integral given by the last line. The introduction of intrinsic worldline gravity via the worldline metric g_{tt} , while not essential, is useful for the generalization to string theory. The worldline metric g_{tt} acts as a Lagrange multiplier that enforces the constraint

$$T_{tt} = (\partial_t X)^2 - g_{tt} m^2 = 0 ; \quad (2)$$

apart from this constraint, the dynamics of worldline gravity is trivial. Indeed, we can fix a gauge $g_{tt} = T$,³ and after rescaling $\tau = Tt$, equation (1) boils down to the standard path integral representation

$$G(x, x') = \int_{\substack{X(0)=x \\ X(T)=x'}} \mathcal{D}X \exp \left[i \int_0^T d\tau [(\partial_\tau X)^2 + m^2] \right] . \quad (3)$$

We can generalize this construction in several ways. For instance, we can put the particle in a curved spacetime with metric $G_{\mu\nu}(X)$, and in a background potential $V(X)$ that generalizes the constant m^2 ;⁴ also, we can couple a charged particle to a background electromagnetic field specified by the vector potential $A_\mu(X)$. The effect is to replace the free particle action in (1) by a generalized ‘worldline nonlinear sigma model’

$$SS_{\text{worldline}} = \int dt \left[\sqrt{g} g^{tt} G_{\mu\nu}(X) \partial_t X^\mu \partial_t X^\nu + A_\mu(X) \partial_t X^\mu - \sqrt{g} V(X) \right] . \quad (4)$$

String theory introduces a second generalization, replacing the notion of dynamics of pointlike objects to that of extended objects such as a one-dimensional string. Perturbative string dynamics is governed by an action which is the analogue of (4)

$$SS_{\text{ws}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\left(\sqrt{g} g^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu + \alpha' \sqrt{g} R^{(2)} \Phi(X) + \sqrt{g} V(X) \right] \quad (5)$$

where $a, b = 0, 1$ and $\mu, \nu = 0, \dots, D-1$ are worldsheet and target space indices, respectively. The quantity $\alpha' = \ell_s^2$ sets a length scale for the target space parametrized by X^μ ; it plays the role of \hbar for the generalized nonlinear sigma model (5). The anti-symmetric tensor gauge field $B_{\mu\nu}$ is the direct generalization of the vector potential A_μ ; the former couples to the area element $dX^\mu \wedge dX^\nu$ of the two-dimensional string worldsheet in the same way that the latter couples to the line element dX^μ of the

³Infinitesimal reparametrizations $\delta g_{tt} = \partial_t v_t$ must fix the endpoints of the parameter space, *i.e.* $v_t = 0$ at $t = 0, 1$. A consequence is that the constant mode of g_{tt} cannot be gauged away, but everything else in g_{tt} can be fixed, allowing the choice $g_{tt} = T$; the integral over metrics modulo reparametrizations thus reduces to the ordinary integral $\int dT$. An analogous phenomenon occurs in the string path integral; the analogues of the parameter T are the *moduli* of the string worldsheet.

⁴Note that m^2 can be thought of as a worldline cosmological constant.

particle worldline. In addition, because intrinsic curvature $R^{(2)}$ can be non-trivial in two dimensions, one has an additional coupling of the curvature density to a field Φ known as the *string dilaton*.

The dynamical principle of the worldsheet theory is the requirement that

$$\langle \cdots T_{ab} \cdots \rangle = 0 \quad (6)$$

in all correlation functions. The two traceless components of these equations play the same role as the constraint (2) – they enforce reparametrization invariance on the worldsheet. The trace component is a requirement that the 2d QFT of the worldsheet dynamics is *locally scale invariant*, *i.e.* that the beta functions vanish. For example, setting $B_{\mu\nu} = V = 0$, the conditions through one loop are

$$\begin{aligned} \beta_{G_{\mu\nu}} &= \alpha'(\mathcal{R}_{\mu\nu}(G) + \nabla_\mu \nabla_\nu \Phi) + O(\alpha'^2) = 0 \\ \beta_\Phi &= \frac{D-26}{6} + \alpha'(\frac{1}{2}\nabla^2 \Phi + (\nabla \Phi)^2) + O(\alpha'^2) = 0 \end{aligned} \quad (7)$$

where $\mathcal{R}_{\mu\nu}(G)$ is the Ricci curvature of the spacetime metric G , and ∇ is the space-time gradient. Thus, a reason to be interested in string theory is that, in contrast to the point particle, the string carries with it the information about what spacetimes it is allowed to propagate in – namely, those that satisfy the Einstein equations coupled to a scalar dilaton (and other fields, if we had kept them nonzero).

Since the local invariances combine the reparametrization group *Diff* and the group of local scale transformations *Weyl*, the appropriate replacement for (1) is

$$\mathcal{Z} = \int \frac{\mathcal{D}g \mathcal{D}X}{\text{Diff} \times \text{Weyl}} \exp[iSS_{\text{ws}}] . \quad (8)$$

We can soak up the local gauge invariance by (locally on the worldsheet) choosing coordinates in which $g_{ab} = \delta_{ab}$. One cannot choose such flat coordinates globally, however, as one sees from the Gauss-Bonnet identity $\int \sqrt{g} R^{(2)} = 4\pi(2 - 2h)$.⁵ Nevertheless, one can relate any metric via the symmetries to one of a $6h - 6$ parameter family of reference metrics $\hat{g}_{ab}(m_r)$, $r = 1, \dots, 6h - 6$. The parameters m_r are called the *moduli* of the 2d surface.⁶ A simple picture of these parameters is shown in figure 1.

Thus, after fixing all of the reparametrization and local scale invariance, the integration over metrics $\int \frac{\mathcal{D}g}{\text{Diff} \times \text{Weyl}}$ reduces to an integration over these moduli. The moduli are the string version of the Schwinger parametrization of the propagator (1) for a particle.

⁵It is standard practice to Wick rotate to Euclidean worldsheets and ignore any associated subtleties. We will follow the standard practice here.

⁶There are a few special cases; for $h = 0$, there are no moduli, and for $h = 1$ there are two moduli (the length and twist of the propagator tube joined to itself to make a torus).

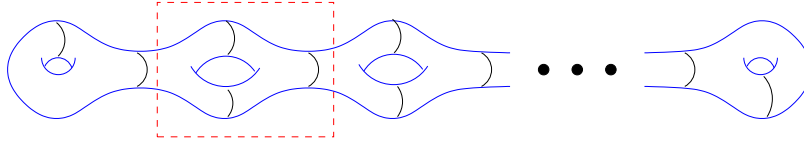


Figure 1: *Each handle, except the end ones, contributes three closed string propagator tubes to the surface. Each tube has a length and a helical twist angle. The two end handles together contribute only three tubes, and so the number of moduli is $6h - 6$.*

3 Strings in D-dimensional spacetime

A simple solution to the equations (7) uses ‘conformally improved’ free fields:⁷

$$G_{\mu\nu} = \eta_{\mu\nu} \quad , \quad B_{\mu\nu} = V = 0 \quad , \quad \Phi = n_\mu X^\mu \quad \left(n^2 = \frac{26-D}{6\alpha'} \right) \quad . \quad (9)$$

The geometry seen by propagating strings is flat spacetime, with a linear dilaton. The dilaton slope is timelike for $D > 26$ and spacelike for $D < 26$.

Just as the perturbative series for particles is a sum over Feynman graphs, organized in order of increasing number of loops in the graph, the perturbative expansion for strings is organized by the number of handles of the corresponding sum over worldsheets, weighted by the effective coupling g_{eff} to the power $2h - 2$ (where h is the number of handles, often called the *genus* of the surface).

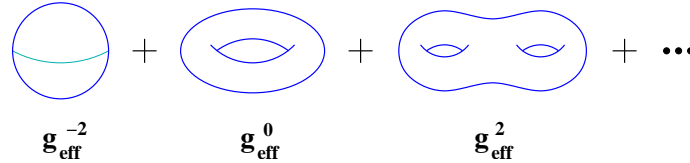


Figure 2: *The string loop expansion.*

Consider string worldsheets in the vicinity of the target space location \hat{X} . Using the Gauss-Bonnet identity, the term

$$\frac{1}{4\pi} \int \sqrt{g} R^{(2)} \Phi(X) \sim \Phi(\hat{X})(2 - 2h) \quad (10)$$

⁷Conformally improved means that while the path integral (8) is Gaussian, so the worldsheet QFT is free, the stress tensor is modified due to the coupling to intrinsic curvature.

in the path integral over the (Euclidean) worldsheet action identifies the effective coupling as

$$g_{\text{eff}} = \exp[\Phi(\hat{X})] . \quad (11)$$

Thus we have strong coupling at large $\Phi = n \cdot X$, and we have to say what happens to strings that go there.

There is also a perturbative instability of the background. Perturbations of the spacetime background are scaling operators. Maintaining conformal invariance at the linearized level imposes marginality of the scaling operator. These marginal scaling operators are known as *vertex operators*. Consider for instance adding the potential term

$$V(X) = \int \frac{d^D k}{(2\pi)^D} v_k e^{ik \cdot X} \quad (12)$$

to the worldsheet action. The scale dimension of an individual Fourier component is determined by its operator product with the stress tensor⁸

$$T(z) e^{ik \cdot X(w)} \underset{z \rightarrow w}{\sim} \frac{\Delta}{(z-w)^2} e^{ik \cdot X(w)} . \quad (13)$$

Using the improved stress tensor⁹

$$T(z) = -\frac{1}{\alpha'} \partial_z X \cdot \partial_z X + n \cdot \partial_z^2 X \quad (14)$$

and evaluating the operator product expansion (13) via Wick contraction with the free propagator

$$X(z)X(w) \sim -\frac{\alpha'}{2} \log |z-w|^2 , \quad (15)$$

one finds the scale dimension

$$\Delta = \frac{\alpha'}{4} k^2 + \frac{i\alpha'}{2} n \cdot k \quad . \quad (16)$$

Thus the condition of linearized scale invariance $\Delta = \bar{\Delta} = 1$ is a mass-shell condition for $V(X)$. This result should be no surprise – local scale invariance gives the equations of motion (7) of the background, so the linearized scale invariance condition should give the wave equation satisfied by small perturbations. The mass shell condition $\Delta = 1$ amounts to

$$(k + in)^2 = -n^2 - \frac{4}{\alpha'} \quad (17)$$

(recall $n^2 = \frac{26-D}{6\alpha'}$). Thus for $D < 2$, perturbations are “massive”, and the string background is stable. For $D = 2$, the perturbations are “massless”, leading to marginal stability. Finally, for $D > 2$ the perturbations are “tachyonic”, and the

⁸We work in local complex coordinates $z = \sigma^0 + i\sigma^1$.

⁹As advertised, the linear dilaton determines the conformal improvement of $T(z) = T_{zz}$ given by the second term.

background is unstable. The field $V(X)$ is conventionally called the *string tachyon* even though strictly speaking that characterization only applies to $D > 2$.

In the stable regime $D \leq 2$, a static background condensate $V(X)$ “cures” the strong coupling problem.¹⁰ Let $n \cdot X = QX_1$ (recall $n^2 > 0$); then for $D < 2$

$$\begin{aligned} V_{\text{backgd}} &= \mu e^{2bX_1} + \tilde{\mu} e^{2\tilde{b}X_1} \\ \left. \begin{matrix} b \\ \tilde{b} \end{matrix} \right\} &= \frac{Q}{2} \mp \sqrt{\left(\frac{Q}{2}\right)^2 - \frac{1}{\alpha'}} = \frac{\sqrt{26-D} \mp \sqrt{2-D}}{\sqrt{24\alpha'}} . \end{aligned} \quad (18)$$

(note that $\tilde{b} = (b\alpha')^{-1}$). For $D = 2$ one has $b = \tilde{b} = 1/\sqrt{\alpha'}$, and so the two exponentials are not independent; rather

$$V_{\text{backgd}}^{(D=2)} = \mu X_1 e^{2bX_1} + \tilde{\mu} e^{2bX_1} . \quad (19)$$

The exponential barrier self-consistently keeps perturbative string physics away from strong coupling for sufficiently large μ .

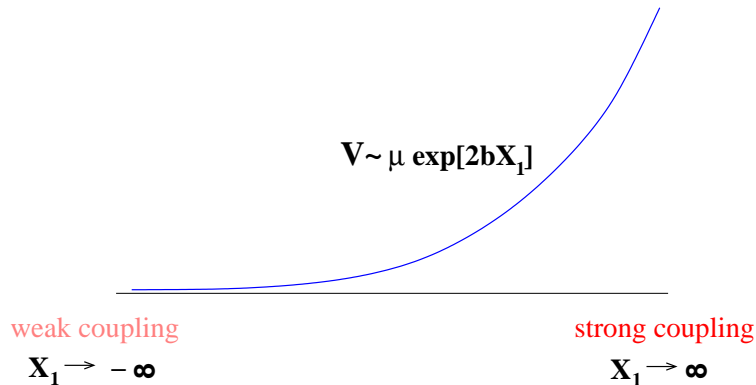


Figure 3: *The tachyon background.*

For example, consider the scattering of a string tachyon of energy E in $D = 2$. The string is a perturbation $\delta V(X) = \exp[-iEX^0 + ikX^1]$, with $ik = \pm iE + Q$ the solution to the on-shell condition $\Delta = 1$. The scattering is depicted in figure 4.

The worldsheet energy $\mathcal{E}_{\text{ws}} = \alpha' E^2/2$ of the zero mode motion in X^1 of the string is determined by the stress tensor $T(z)$, equation (14); it is essentially the X^1

¹⁰One should worry whether the conformal invariance condition is satisfied beyond the linearized level when one promotes the tachyon V from a perturbation to a full term in the action describing string propagation. Fortunately, conformal invariance in the presence of the exponential interaction was demonstrated by operator methods in [27]. The issue is rendered moot by the construction of a conformal bootstrap (an ansatz for the correlation functions of the exact theory) [10, 11], which we sketch below in section 6.2.

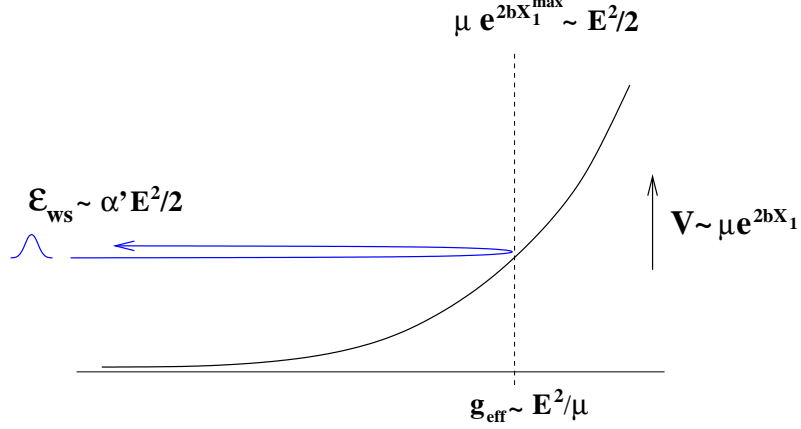


Figure 4: *Scattering of a tachyon excitation off the tachyon background.*

contribution to Δ . The turning point of the motion is determined by this energy to be

$$V_{\text{backgd}}(X_1^{\text{max}}) = \mathcal{E}_{\text{ws}} = \alpha' E^2/2 . \quad (20)$$

The effective coupling is largest at this point,

$$g_{\text{eff}} \sim e^{\Phi(X_1^{\text{max}})} \sim E^2/\mu ; \quad (21)$$

thus low energy scattering is self-consistently weakly coupled. The effective coupling is determined by the value of the dilaton at the turning point; we expect the scattering amplitude to have a perturbative series in powers of E^2/μ . Note that the high energy behavior is nonperturbative, however.

At this point we choose to relabel for $D = 2$ the spacetime coordinates

$$\phi \equiv X^1 \quad , \quad X \equiv X^0 \quad (22)$$

in order to conform to standard notation in the subject, as well as to reduce the clutter of indices. Also, we will henceforth set $\alpha' = 1$ as a choice of units (*i.e.* we measure all spacetime lengths in “string units”).

3.1 A reinterpretation of the background

The 2d QFT¹¹ of the “tachyon” background

$$SS_{\text{ws}} = \frac{1}{4\pi} \int \sqrt{g} \left[g^{ab} \partial_a \phi \partial_b \phi + bQ R^{(2)} \phi + \mu e^{2b\phi} \right] \quad (23)$$

¹¹In an attempt to reduce confusion, the notation 2d will be used when referring to the string worldsheet dimension, while 2D will refer to two-dimensional spacetime backgrounds.

has an alternative interpretation in terms of worldsheet intrinsic geometry [28], where $e^{2b\phi}g_{ab}$ is interpreted as a *dynamical metric*, and the remaining $D - 1$ fields X are thought of as “matter” coupled to this dynamical gravity. Let $\varphi = b\phi$; then the action becomes

$$SS_{\text{ws}} = \frac{1}{4\pi b^2} \int \sqrt{g} \left[(\nabla\varphi)^2 + bQ R^{(2)}\varphi + \mu b^2 e^{2\varphi} \right]. \quad (24)$$

Note that b plays the role of the coupling constant; the semi-classical limit is $b \rightarrow 0$ (and thus $Q = b^{-1} + b \rightarrow b^{-1}$). The equation of motion for φ reads

$$\nabla_g^2 2\varphi - bQ R^{(2)}[g] = 2\mu b^2 e^{2\varphi}. \quad (25)$$

Here ∇_g is the covariant derivative with respect to the intrinsic metric g_{ab} . Due to the properties of the curvature under local rescaling,

$$\nabla_g^2 2\varphi - R^{(2)}[g] = -e^{2\varphi} R^{(2)}[e^{2\varphi}g], \quad (26)$$

the combination on the left-hand side of (25) is, in the semi-classical limit $b \rightarrow 0$, just the curvature of the dynamical metric $e^{2\varphi}g_{ab}$. The equation of motion can be written as the condition for constant curvature of this dynamical metric

$$R^{(2)}[e^{2\varphi}g] = -2\mu b^2, \quad (27)$$

known as the *Liouville equation*; the theory governed by the action (24) is the Liouville field theory. The equation (25) is the appropriate quantum generalization of the Liouville equation. The constant on the right-hand side of (27) is a cosmological constant for the 2d intrinsic fluctuating geometry.¹² Note that $\sqrt{g}e^{2\varphi}$ is the “dynamical area element”, so that the potential term in the action (24) is a chemical potential for the dynamical intrinsic area of the worldsheet.

This interpretation of the static tachyon background in terms of fluctuating intrinsic geometry is only available for $D \leq 2$. For $D > 2$, the on-shell condition $\Delta = 1$ (equation (17)) is not solved by $V_{\text{backgd}} = e^{2bX_1}$ for real b (rather $b = \frac{1}{2}Q \pm i\lambda$), and so $\sqrt{g}V_{\text{backgd}}$ is not the area of a dynamical surface.

3.2 KPZ scaling

The fact that the dynamical metric is integrated over yields useful information about the scaling of the partition and correlation functions with respect to the cosmological constant μ , known as *KPZ scaling* [29, 30, 31]. Consider the shift $\varphi \rightarrow \varphi + \frac{\epsilon}{2}$ in the Liouville action (24) in genus h ; this leads to

$$SS_h(\mu) \longrightarrow SS_h(e^\epsilon \mu) + (2 - 2h) \frac{Q}{2b} \epsilon. \quad (28)$$

¹²The factor of b^2 on the right-hand side can be absorbed into a renormalization of μ , so that the semi-classical limit is well-behaved.

However, this constant mode of φ is integrated over in the Liouville partition function, and therefore $\mathcal{Z}_h(\mu)$ must be independent of ϵ . We conclude

$$\mathcal{Z}_h(\mu) = \mathcal{Z}_h(e^\epsilon \mu) \exp[-(2-2h)\frac{Q}{2b}\epsilon] \implies \mathcal{Z}_h(\mu) = c_h \mu^{(2-2h)Q/2b}. \quad (29)$$

For instance, for $D = 1$ (pure Liouville gravity, with no matter) one finds $\frac{Q}{2b} = \frac{5}{4}$, and so the genus expansion of the partition function is a series in $\mu^{-5/2}$. For $D = 2$, we have $\frac{Q}{2b} = 1$, and so the partition function is a series in μ^{-2} .^{13,14}

We could now pass to a discussion of correlation functions of this 2d Liouville QFT, and their relation to the scattering of strings. Instead, we will suspend this thread of development in favor of a random matrix formulation of the same physics. We will return to the quantization of Liouville theory later, when it is time to forge the link between these two approaches.

4 Discretized surfaces and 2D string theory

For spacetime dimension $D \leq 2$, we have arrived at an interpretation of the path integral describing string propagation in the presence of a background tachyon condensate as a sum over dynamical worldsheet geometries, in the presence of $D - 1$ “matter fields”.¹⁵

A discrete or lattice formulation of fluctuating worldsheet geometry can be given in terms of matrix Feynman graphs. Any tessellation of a surface built of regular polygons (see figure 5 for a patch of tessellated surface) has a dual¹⁶ double-line “fatgraph”, also depicted in figure 5. The double lines indicate the flow of matrix index contractions around the graph.

¹³In applying the KPZ scaling argument, one needs to be sure that the path integral over the zero mode of φ is convergent. This is the case for $h > 1$, where the Gauss-Bonnet theorem tells us there is a classical solution to Liouville theory since the mean curvature is negative; expanding around a metric g_{ab} of constant negative curvature, the effective potential for φ has a stable minimum due to the competition between the exponential potential and the linear $R^{(2)}\varphi$ term. For $h = 0, 1$, this linear term is either absent or pushes the wrong way; there is no local minimum for φ , and the path integral over the zero mode diverges. In fact, one can show that this difficulty occurs whenever the power of μ predicted by KPZ scaling is non-negative. In these cases, one can take enough derivatives with respect to μ (*i.e.* a correlation function of several operators $e^{2b\varphi}$) so that the scaling argument applies, and then recover the partition function by integrating back up. For $D = 1$ this results in logarithmic corrections to KPZ scaling, since $Q/2b$ is integral.

¹⁴The linear term in (19) leads to subtleties in the application of KPZ scaling, see [32, 8, 9].

¹⁵In solving the local scale invariance condition $\beta_\Phi = 0$ for the string dilaton, $D - 1$ is the contribution of the fields other than $\varphi = X^1/b$ to the leading term, the so-called *conformal anomaly* or *conformal central charge*

$$\langle T_a^a(\text{matter}) \rangle = \frac{c_{\text{matter}}}{48\pi} R^{(2)}(g)$$

with $c_{\text{matter}} = D - 1$ when the remaining theory is conformal. This contribution is then cancelled by the Liouville QFT, together with a contribution -26 from the reparametrization Faddeev-Popov ghosts. This formally allows us to consider fractional D (and even $D < 0$ if we allow non-unitary matter CFT’s), through the use of interacting CFT’s of appropriate c_{matter} .

¹⁶In the sense of Poincaré.

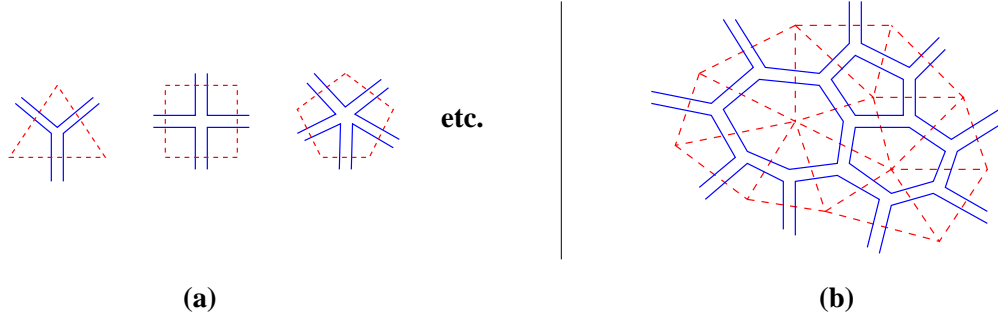


Figure 5: (a) Regular polygons for tiling a surface, with dashed red edges; and the dual fatgraph vertices, with solid blue dual edges. (b) A patch of discrete surface tessellated with triangles, and the dual fatgraph.

The partition function

$$\begin{aligned}\mathcal{Z}(g_i) &= \int \mathcal{D}^{N^2} M \exp[-\text{tr}(\frac{1}{2}g_2 M^2 + \mathcal{U}(M))] \\ \mathcal{U}(M) &= \frac{1}{3}g_3 M^3 + \frac{1}{4}g_4 M^4 + \dots\end{aligned}\quad (30)$$

serves as a generating function for fatgraphs, and thereby defines an ensemble of random surfaces. For example, consider a surface with triangles only, $g_{i>4} = 0$. Each face of the fatgraph gives a factor N from the trace over the index loop bordering the face. Each vertex gives a factor g_3 , and each propagator $1/g_2$. The partition function

$$\mathcal{Z}(g) = \sum_{V,E,F} (g_3)^V (1/g_2)^E N^F d(V, E, F) \quad (31)$$

sums over the number $d(V, E, F)$ graphs with V vertices, E edges (propagators), and F faces. Using the fact that each propagator shares two vertices, and each vertex ends three propagators, one has $2E = 3V$. The discrete version of the Gauss-Bonnet theorem (the Euler identity) is $V - E + F = 2 - 2h$. The partition function is thus

$$\mathcal{Z}(g) = \sum_{h=0}^{\infty} \sum_A N^{2-2h} \left(\frac{g_3 N^{1/2}}{g_2^{3/2}} \right)^A d(h, A) \quad (32)$$

where here and hereafter we write $V = A$, since the number of vertices A is the discrete area of the surface. Large N thus controls the topological expansion: $g_s^{\text{discrete}} = 1/N$ is the string coupling of the discrete theory. The cosmological constant of the discrete theory is the free energy cost of adding area (triangles): $\mu^{\text{discrete}} = -\log(g_3 N^{1/2}/g_2^{3/2})$.

Being a lattice theory, in order to compare with the continuum formulation of previous sections we need to take the continuum limit of the matrix integral. That

is, we want to send the discrete area A to infinity in units of the lattice spacing (or equivalently, send the lattice spacing to zero for a “typical” surface in the ensemble).

Taking this limit amounts to balancing the suppression of surface area by the 2d cosmological constant μ_{discrete} against the entropy $d(h, A)$ of large Feynman graphs (roughly, if we want to add an extra vertex to a planar graph, there are of order A places to put it). In other words, one searches for a phase transition or singularity in $\mathcal{Z}(g)$ where for some g_{crit} the partition sum is dominated by graphs with an asymptotically large number of vertices. Universality of this kind of critical phenomenon is the statement that the critical point is largely independent of the detailed form of the matrix potential $\mathcal{U}(M)$, for instance whether the dual tessellation uses triangles or squares in the microscopic theory (*i.e.* M^3 vs. M^4 interaction vertices in the graphical expansion).

Before discussing this phase transition, let us add in the matter. We wish to put discretized scalar field theory on the random surfaces generated by the path integral over M . The following modification does the job:

$$\mathcal{Z} = \int \mathcal{D}M \exp \left[\text{tr} \left(\int dx \int dx' \frac{1}{2} M(x) G^{-1}(x - x') M(x') + \int dx \mathcal{U}(M(x)) \right) \right]. \quad (33)$$

In the large N expansion, we now have a propagator $G(x - x')$ in the Feynman rules (rather than $g_2^{-1} = \text{const.}$). Thus, on a given graph we have a product of propagators along the edges

$$\prod_{\text{edges}} (\text{propagators}) = \prod_{\substack{i,j \\ \text{neighbors}}} G(x_i - x_j); \quad (34)$$

the choice $G(x - x') = \exp[-(x - x')^2/\beta]$ leads to the discretized kinetic energy of a scalar field X

$$\prod_{\substack{i,j \\ \text{neighbors}}} G(x_i - x_j) = \exp \left[-\frac{1}{\beta} \sum_{\substack{i,j \\ \text{nghbrs}}} (x_i - x_j)^2 \right] \quad (35)$$

which is the appropriate path integral weight for a scalar field on the lattice. The evaluation of the graph involves an integral $\prod_i \int dx_i$ over the location in x -space of all the vertices. In other words, we path integrate over the discretized scalar field with the probability measure (35).

Unfortunately, the gaussian kinetic energy that leads to this form of the propagator is not standard. Fortunately, for $D = 2$ (*i.e.* one scalar matter field) the choice $G(x - x') = \exp[-|x - x'|/\beta^{1/2}]$ turns out to be in the same universality class, and arises from a canonical kinetic energy for the matrix path integral

$$\begin{aligned} G^{-1}(p) = e^{\beta p^2} &\longleftrightarrow G(x) = \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} e^{-x^2/4\beta} \\ \tilde{G}^{-1}(p) = 1 + \beta p^2 &\longleftrightarrow \tilde{G}(x) = \frac{\pi}{\sqrt{\beta}} e^{-|x|/\beta^{1/2}} \end{aligned} \quad (36)$$

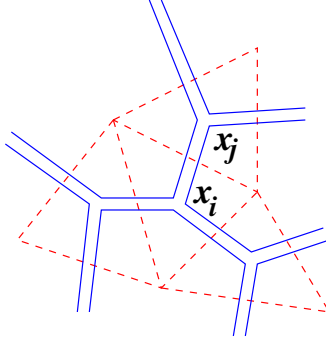


Figure 6: *Fatgraph vertices live at points in x space. The product over propagators weights the sum over configurations $\{x_i\}$ by a nearest-neighbor interaction determined by the propagator $G(x_i - x_j)$.*

The continuum limit involves scalar field configurations which are slowly varying on the scale of the lattice spacing, which is enforced by taking $\beta p^2 \rightarrow 0$. But in this limit $G^{-1} \sim \tilde{G}^{-1}$ and so we expect the two choices to lead to the same continuum physics. But in $D = 2$ (*i.e.* one-dimensional x -space), $\tilde{G}(p)$ is the conventional Feynman propagator for M , and so we may write

$$\mathcal{Z} = \int \mathcal{D}M \exp \left\{ - \int dx \operatorname{tr} \left[\frac{\beta}{2} \left(\frac{dM}{dx} \right)^2 - \mathcal{U}(M) \right] \right\}, \quad (37)$$

now with $\mathcal{U}(M) = -\frac{1}{2}M^2 - \frac{1}{3}gM^3$.

To analyze this path integral, it is most convenient to use the matrix analogue of polar coordinates. That is, let

$$M(x) = \Omega(x) \Lambda(x) \Omega^{-1}(x) \quad (38)$$

where $\Omega \in U(N)$ and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. The integration measure $\mathcal{D}M$ becomes in these variables

$$\mathcal{D}M = \mathcal{D}\Omega \mathcal{D}\Lambda \Delta^2(\Lambda) \quad , \quad \Delta(\Lambda) = \prod_{i < j} (\lambda_i - \lambda_j) \quad (39)$$

where $\mathcal{D}\Omega$ is the $U(N)$ group (Haar) measure.

A useful intuition to keep in mind is the analogous transformation from Cartesian to spherical coordinates for integration over the vector space \mathbb{R}^n . One uses the rotational invariance of the measure to write $d^n x = d\Omega_{n-1} dr r^{n-1}$, with Ω_{n-1} the space of angles which parametrize an orbit under the rotational group $O(n)$; r parametrizes which orbit we have, and r^{n-1} is the size of the orbit. The orbits degenerate at the origin $r = 0$, due to its invariance under $O(n)$, and this degeneration is responsible

for the vanishing of the Jacobian factor r^{n-1} on this degenerate orbit. Similarly, in the integration over matrices \mathcal{DM} is the Cartesian measure on the matrix elements of M . The invariance of this measure under unitary conjugation of M allows us to pass to an integration over $U(N)$ orbits, parametrized by the diagonal matrix of eigenvalues Λ . The (*Vandermonde*) Jacobian factor $\Delta^2(\Lambda)$ characterizes the size of an orbit; the orbits degenerate whenever a pair of eigenvalues coincide, since the action of $SU(2) \subset U(N)$ (that rotates these eigenvalues into one another) degenerates at such points. The overall power of the Vandermonde determinant is determined by scaling (just as the power r^{n-1} is fixed for the vector measure).

In these variables, the Hamiltonian for the matrix quantum mechanics (37) is

$$H = \sum_i \left[-\frac{\beta}{2} \frac{1}{\Delta^2} \frac{\partial}{\partial \lambda_i} \Delta^2 \frac{\partial}{\partial \lambda_i} + \mathcal{U}(M) \right] + \frac{1}{2\beta} \sum_{i < j} \frac{\hat{\Pi}_{ij} \hat{\Pi}_{ji}}{(\lambda_i - \lambda_j)^2} \quad (40)$$

where $\hat{\Pi}_{ij}$ is the left-invariant momentum on $U(N)$, and the ordering has been chosen so that the operator is Hermitian with respect to the measure (39). The last term is the analogue of the angular momentum barrier in the Laplacian on \mathbb{R}^n in spherical coordinates. Note that the kinetic operator for the eigenvalues can be rewritten

$$\sum_i \frac{1}{\Delta^2} \frac{\partial}{\partial \lambda_i} \Delta^2 \frac{\partial}{\partial \lambda_i} = \sum_i \frac{1}{\Delta} \frac{\partial^2}{\partial \lambda_i^2} \Delta. \quad (41)$$

Wavefunctions for the $U(N)$ angular degrees of freedom will transform in representations of $U(N)$. The simplest possibility is to choose the trivial representation, $\Psi_{U(N)}(\Omega) = 1$. In this $U(N)$ singlet sector, we can write the wavefunction as

$$\Psi(\Omega, \Lambda) = \Psi_{\text{eval}}(\Lambda) = \Delta^{-1}(\Lambda) \tilde{\Psi}(\Lambda) \quad (42)$$

and the Schrödinger equation becomes [33]

$$H \Psi_{\text{eval}}(\Lambda) = \Delta^{-1}(\Lambda) \sum_i \left[-\frac{\beta}{2} \frac{\partial^2}{\partial \lambda_i^2} + \mathcal{U}(\lambda_i) \right] \tilde{\Psi}(\Lambda), \quad (43)$$

i.e. the eigenvalues are *decoupled* particles moving in the potential $\mathcal{U}(\lambda)$. The wavefunction Ψ_{eval} is symmetric under permutation of the eigenvalues in the $U(N)$ singlet sector (these permutations are just the Weyl group action of $U(N)$); consequently $\tilde{\Psi}$ is totally antisymmetric under eigenvalue permutations – *the eigenvalues behave effectively as free fermions*.

4.1 An aside on non-singlets

What about non-singlet excitations? Gross and Klebanov [34, 8, 35] estimated the energy cost of non-singlet excitations and found it to be of order $O(-\log \epsilon)$, where

$\epsilon \rightarrow 0$ characterizes the continuum limit. Hence, angular excitations decouple energetically in the continuum limit. Alternatively, one can gauge the $U(N)$, replacing $\partial_x M$ by the covariant derivative $D_x M = \partial_x M + [A, M]$; the Gauss law of the gauge theory then projects onto $U(N)$ singlets.

The physical significance of non-singlet excitations is exhibited if we consider the theory in periodic Euclidean time $x \in \mathbb{S}^1$, $x \sim x + 2\pi R$, appropriate to the computation of the thermal partition function. In the matrix path integral, we must allow twisted boundary conditions for M [34, 8, 35]:

$$M(x + 2\pi R) = \Omega M(x) \Omega^{-1} \quad , \quad \Omega \in U(N) \quad . \quad (44)$$

The matrix propagator is modified to

$$\langle M_i^k(x) M_j^l(x') \rangle = \sum_{m=-\infty}^{\infty} e^{-|x-x'+2\pi Rm|} (\Omega^m)_i^l (\Omega^{-m})_j^k . \quad (45)$$

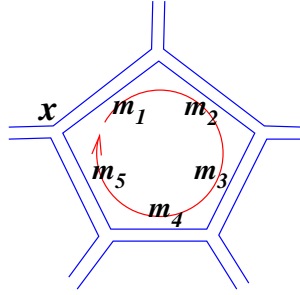


Figure 7: *The product over twisted propagators around the face of a fatgraph allows monodromy for x , corresponding to a vortex insertion.*

Consider a fixed set of $\{m_i\}$ and a fixed fatgraph. Following the propagators along the index line that bounds the face of a planar graph, figure 7, we see that the coordinate of a fatgraph vertex along the boundary shifts by

$$x \longrightarrow x + 2\pi \left(\sum_i m_i \right) R ; \quad (46)$$

thus the sum over $\{m_i\}$ is a sum over vortex insertions on the faces of the graph (the vertices of the dual tessellation). The sum over twisted boundary conditions introduces vortices into the partition sum for the scalar matter field X . We can now understand the suppression of non-singlet wavefunctions as a reflection of the suppression of vortices in the 2d QFT of a periodic scalar below the Kosterlitz-Thouless transition.

4.2 The continuum limit

We are finally ready to discuss the continuum limit of the sum over surfaces. Recall that we wish to take $N \rightarrow \infty$, with the potential tuned to the vicinity of a phase transition – a nonanalytic point in the free energy as a function of the couplings in the potential $\mathcal{U}(M)$. We now know that the dynamics is effectively that of free fermionic matrix eigenvalues, moving in the potential $\mathcal{U}(\lambda)$. Consider $\mathcal{U}(\lambda) = -\frac{1}{2}\lambda^2 - g\lambda^3$, figure 8a.

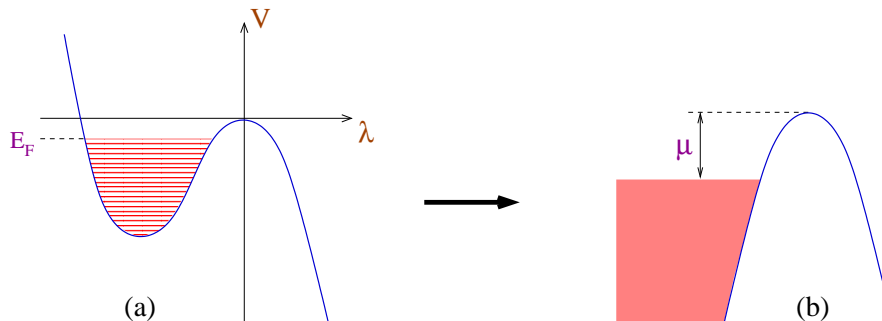


Figure 8: (a) Cubic eigenvalue potential. For small g , there are many metastable levels. (b) The scaling limit focusses on the vicinity of the local maximum of the potential.

There are many metastable levels in the well on the left of the local maximum of the potential. The coupling g can be tuned so that there are more than N such metastable single-particle states. As N is sent to infinity, one can adjust $g \rightarrow 0$ so that there are always N levels in the well. The metastable Fermi energy E_F will be a function of g and N . Consider an initial state where these states are populated up to some Fermi energy E_F below the top of the barrier, and send $g \rightarrow 0$, $N \rightarrow \infty$, such that $E_F \rightarrow 0^-$. In other words, the phase transition we seek is the point where eigenvalues are about to spill over the top of the potential barrier out of the well on the left. The resulting situation is depicted in figure 8b, where we have focussed in on the quadratic maximum of the potential via the rescaling $\hat{\lambda} = \lambda/\sqrt{N}$, so that $\mathcal{U}(\hat{\lambda}) \sim -\frac{1}{2}\hat{\lambda}^2$. We hold $\mu = -NE_F$ fixed in the limit. The result is quantum mechanics of free fermions in an inverted harmonic oscillator potential, with Fermi level $-\mu < 0$. To avoid notational clutter, we will drop the hat on the rescaled eigenvalue, continuing to use λ as the eigenvalue coordinate even though it has been rescaled by a factor of \sqrt{N} from its original definition.

A useful perspective on the phase transition comes from consideration of the classical limit of the ensemble of eigenvalue fermions. The leading semiclassical approximation to the degenerate Fermi fluid of eigenvalues describes it as an incompressible fluid in phase space [36, 37]. Each eigenvalue fermion occupies a cell

of volume $2\pi\hbar$ in phase space, with one fermion per cell; the classical limit is a continuous fluid, which is incompressible due to Pauli exclusion. The metastable ground state, which becomes stable in this limit, has the fluid filling the interior of the energy surface in phase space of energy E_F ; see figure 9.

The universal part of the free energy comes from the endpoint of the eigenvalue distribution near $\lambda \sim 0$. The limit $E_F \rightarrow 0^-$ leads to a change in this universal component, due to the singular endpoint behavior $\rho(\lambda) \sim \sqrt{\lambda^2 - E_F}$ of the eigenvalue density in this limit.

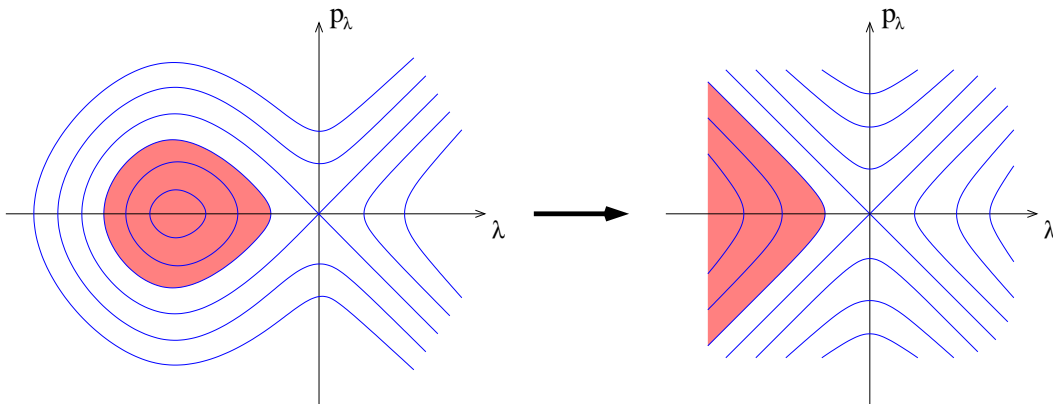


Figure 9: *Phase space portrait of the classical limit of the free fermion ground state. The contours are orbits of fixed energy; the shaded region depicts the filled Fermi sea.*

One should worry that the theory we have described is not well-defined, due to the fact that there is a finite rate of tunnelling of eigenvalues out of the metastable well. Single-particle wavefunctions in the inverted harmonic potential are parabolic cylinder functions

$$\psi_\omega(\lambda) = c_\omega D_{-\frac{1}{2}+i\omega}((1+i)\lambda) \stackrel{\lambda \rightarrow \infty}{\sim} \frac{1}{\sqrt{\pi\lambda}} e^{-i\lambda^2/2+i\omega \log|\lambda|} . \quad (47)$$

If we consider an incoming wave from the left with these asymptotics, with energy $E = -\omega < 0$, a WKB estimate of the tunnelling amplitude gives $T(\omega) \sim e^{-\pi\omega}$. Perturbation theory is an asymptotic expansion in $1/N \propto 1/\mu$ (from KPZ scaling), and since all filled levels have $\omega > \mu$, tunnelling effects behave as e^{-cN} for some constant c and can be ignored if one is only interested in the genus expansion. The genus expansion is the asymptotic expansion around $\mu \rightarrow \infty$, where tunnelling is strictly forbidden.¹⁷ The worldsheet formalism is defined through the genus expansion; ef-

¹⁷This is especially clear in the description of the classical limit as a classical, incompressible fluid in phase space. The classical fluid cannot escape the potential well via tunnelling.

fects such as tunnelling are invisible at fixed genus.¹⁸ Nonperturbatively (at finite μ), the theory does not exist; yet we can make an asymptotic expansion around the metastable configuration of the matrix quantum mechanics, and compare the terms to the results of the worldsheet path integral. We will return to this point in section 8, where the analogous (and nonperturbatively stable) matrix model for the fermionic string is briefly discussed.

The claim is that the continuum limit of the matrix path integral just defined (valid at least in the asymptotic expansion in $1/\mu$) is in the *same universality class* as the $D = 2$ string theory defined via the worldsheet path integral for Liouville theory coupled to $c_{\text{matter}} = 1$ (and Faddeev-Popov ghosts).

5 An overview of observables

Now that we have defined the model of interest, in both the continuum worldsheet and matrix formulations, the next issue concerns the observables of the theory – what physical questions can we ask? In this section we discuss three examples of observables: (i) macroscopic loop operators, which put holes in the string worldsheet; (ii) asymptotic scattering states, the components of the S-matrix; and (iii) conserved charges, which are present in abundance in any free theory (*e.g.* the energies of the particles are separately conserved).

5.1 Loops

Consider the matrix operator

$$\begin{aligned} W(\mathbf{z}, x) &= -\frac{1}{N} \text{tr}[\log(\mathbf{z} - M(x))] \\ &= +\frac{1}{N} \sum_{l=1}^{\infty} \frac{1}{l} \text{tr}[(M(x)/\mathbf{z})^l] - \log \mathbf{z} . \end{aligned} \quad (48)$$

From the matrix point of view, $\exp[W(\mathbf{z}, x)] = \det[\mathbf{z} - M(x)]$ is the characteristic polynomial of $M(x)$, and thus a natural collective observable of the eigenvalues. Note that \mathbf{z} parametrizes the eigenvalue coordinate. As a collective observable of the matrix, this operator is rather natural – its exponential is the characteristic polynomial of the matrix $M(x)$, and hence encodes the information contained in the distribution of matrix eigenvalues.¹⁹ On a discretized surface, $\frac{1}{l} \text{tr}[M^l(x)]$ is the

¹⁸The existence of tunnelling phenomena is reflected in the genus expansion as the rate of divergence of the contribution of large orders in the expansion [38]; for a discussion in the present context, see [39].

¹⁹It is amusing to note that the same operator appears in the correspondence between $AdS_5 \times S^5$ and maximally supersymmetric gauge theory, as the operator that creates the gauge theory representation of so-called “giant gravitons” in AdS [40].

operator that punches a hole in a surface of lattice length l ; see figure 10.²⁰ All edges bordering the hole are pierced by a propagator which leads to the point x in target space, and the other end of each propagator also goes to the point x in the continuum limit $\beta \rightarrow 0$. Thus the continuum theory has a Dirichlet condition for x along the boundary.

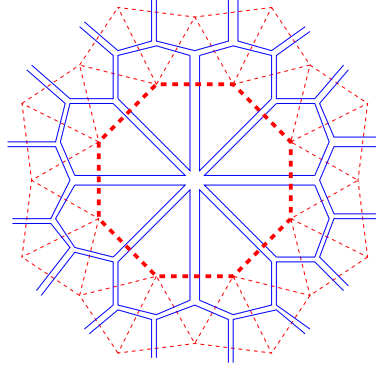


Figure 10: *The operator $\text{tr}[M^l(x)]/l$ inserts a boundary of lattice length l into the fatgraph ($l = 8$ is depicted).*

It is useful to rewrite the loop operator $W(\mathbf{z}, x)$ as follows:

$$\begin{aligned}
W(\mathbf{z}, x) &= -\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \frac{d\ell}{\ell} \frac{1}{N} \text{tr} \exp[-\ell(\mathbf{z} - M(x))] + \log \epsilon \\
&= -\int \frac{d\ell}{\ell} \int d\lambda e^{-\ell(\mathbf{z} - \lambda)} \hat{\rho}(x, \lambda) + \log \epsilon \\
&= -\int \frac{d\ell}{\ell} e^{-\ell \mathbf{z}} \widetilde{W}(\ell, x) + \log \epsilon \quad ,
\end{aligned} \tag{49}$$

where in the first line we have simply introduced an integral representation for the logarithm, while in the second we have rewritten the trace over a function $f(M)$ of the matrix as an integral over the eigenvalue coordinate λ of $f(\lambda)$ times the eigenvalue density operator $\hat{\rho}(x, \lambda)$. This defines the operator in the third line as

$$\widetilde{W}(\ell, x) = \int d\lambda e^{\ell \lambda} \hat{\rho}(x, \lambda) \quad , \tag{50}$$

the Laplace transform of the eigenvalue density operator (recall that classically, the support of ρ is along $(\lambda \in (-\infty, -\sqrt{2\mu}))$). The density operator is a bilinear of the

²⁰The coefficient $1/l$ is a symmetry factor – cyclic rotations of the legs of the vertex $\text{tr}[M^l]$ yield the same fatgraph.

fermion field operator

$$\begin{aligned}\hat{\rho}(x, \lambda) &= \hat{\psi}^\dagger \hat{\psi}(x, \lambda) \\ \hat{\psi}(x, \lambda) &= \int d\nu b_\nu \psi_\nu(\lambda) e^{-i\nu x}\end{aligned}\tag{51}$$

and its conjugate $\hat{\psi}^\dagger$ containing b_ν^\dagger , with the anticommutation relation of mode operators

$$\{b_\nu^\dagger, b_{\nu'}\} = \delta(\nu - \nu') .\tag{52}$$

The mode wavefunctions are given in (47). The operator \widetilde{W} is often called the *macroscopic loop operator*.

In the continuum formalism, we should consider the path integral on surfaces with boundary. The boundary condition on X will be Dirichlet, as discussed above. For the Liouville field ϕ , we use free (Neumann) boundary conditions, but with a boundary interaction

$$SS_L = \frac{1}{4\pi} \int \sqrt{g} [(\nabla_g \phi)^2 + QR^{(2)} + \mu e^{2b\phi}] + \sum_i \oint_{B_i} \mu_B^{(i)} e^{b\phi} .\tag{53}$$

Here, $\oint_{B_i} e^{b\phi} = \ell_{\text{bdy}}^{(i)}$ is the proper length of the i^{th} boundary as measured in the dynamical metric; hence, $\mu_B^{(i)}$ is the *boundary cosmological constant* on that boundary component. The path integral over the dynamical metric sums over boundary lengths with the weight e^{-SS_L} , and therefore produces an integral transform with respect to the lengths of all boundaries. This transform has the same structure as the last line of (49). Let us truncate to zero modes along each boundary component, $\ell_{\text{bdy}}^{(i)} = e^{b\phi_0^{(i)}}$. The path integral measure includes $\int d\phi_0^{(i)} = \int d\ell_{\text{bdy}}^{(i)} / \ell_{\text{bdy}}^{(i)}$, and the weight e^{-SS_L} includes $e^{-\mu_B^{(i)} \ell_{\text{bdy}}^{(i)}} \mathcal{P}(\ell_{\text{bdy}}^{(i)})$, where $\mathcal{P}(\ell_{\text{bdy}}^{(i)})$ is the probability measure for fixed boundary lengths. Comparison with (49) suggests we identify ℓ in $\widetilde{W}(\ell, x)$ as ℓ_{bdy} ; $\mathbf{z} = \mu_B$; and $\mathcal{P}(\ell)$ is the correlator of a product of loop operators $\widetilde{W}(\ell, x)$.

Note in particular that the eigenvalue space of λ , which by (48) is the same as \mathbf{z} -space, is related to ℓ -space (the Liouville coordinate ϕ) by an *integral transform*. They are not the same! However, it is true that asymptotic plane waves in ϕ are the same as asymptotic plane waves in $\log \lambda$.

5.2 The S-matrix

Another observable is the S-matrix. The standard worldsheet prescription for string scattering amplitudes is to evaluate the integrated correlation functions of on-shell vertex operators. Asymptotic tachyon perturbations are produced by the operators

$$V_{i\omega}^{\text{in,out}} = \alpha_\pm(\omega) e^{i\omega(x \mp \phi)} e^{Q\phi}\tag{54}$$

(whose dimension $\Delta = \bar{\Delta} = 1$ follows from (16)). The factor $e^{Q\phi}$ is just the local effective string coupling (11). The vibrational modes of the string are physical only

in directions transverse to the string's worldsheet. Since the worldsheet occupies the only two dimensions of spacetime which are available, there are no transversely polarized string excitations and the only physical string states are the tachyon modes, which have only center-of-mass motion of the string. Actually, this statement is only true at generic momenta. For special momenta, there are additional states (in fact these momenta located at the poles in the relative normalization of $V_{i\omega}^{\text{matrix}}$ and $V_{i\omega}^{\text{continuum}}$). The effects of these extra states are rather subtle; for details, the reader is referred to [9]. The perturbative series for the tachyon S-matrix is

$$S(\omega_i|\omega'_j) = \sum_{h=0}^{\infty} \int \prod_r dm_r \left\langle \prod_i \int d^2z_i V_{i\omega_i}^{(\text{in})} \prod_j \int d^2w_j V_{j\omega'_j}^{(\text{out})} \right\rangle. \quad (55)$$

Actually, the statement that the tachyon is the only physical excitation is only true at generic momenta. For special momenta, there are additional states (in fact these momenta located at the poles in the relative normalization of $V_{i\omega}^{\text{matrix}}$ and $V_{i\omega}^{\text{continuum}}$, see section 6.3). The effects of these extra states are rather subtle; for details and further references, the reader is referred to [9].

In the matrix approach, the *in* and *out* modes are ripples (density perturbations) on the surface of the Fermi sea of the asymptotic form

$$\delta\hat{\rho}(\omega, \lambda) = \hat{\psi}^\dagger \hat{\psi}(\omega, \lambda) \xrightarrow{\lambda \rightarrow -\infty} \frac{1}{2\lambda} \left(\alpha_+(\omega) e^{+i\omega \log |\lambda|} + \alpha_-(\omega) e^{-i\omega \log |\lambda|} \right) \quad (56)$$

as we will verify in the next section. The $\alpha_{\pm}(\omega)$ are right- and left-moving modes of a free field in $x \pm \log |\lambda|$, normalized as

$$[\alpha_{\omega}^{\pm}, \alpha_{\omega'}^{\pm}] = -\omega \delta(\omega + \omega'). \quad (57)$$

Thus, to calculate the S-matrix we should perform a kind of LSZ reduction of the eigenvalue density correlators [41]. Once again, as in the case of the macroscopic loop, the primary object is the density correlator.

The phase space fluid picture of the classical theory leads to an efficient method to compute the classical S-matrix [37, 42], and provides an appealing picture of the classical dynamics of the tachyon field.

5.3 Conserved charges

Since the dynamics of the matrix model is that of free fermions, there will be an infinite number of conserved quantities of the motion. For instance, the energies of each of the fermions is separately conserved. In fact, all of the phase space functions

$$q_{mn}(\lambda, p) = (\lambda + p)^{r-1} (\lambda - p)^{s-1} e^{-(r-s)x} \quad (58)$$

(p is the conjugate momentum to λ) are time independent for motion of a particle in the inverted oscillator potential, generated by $H = \frac{1}{2}(p^2 - \lambda^2)$, ignoring operator ordering issues. These charges generate canonical transformations, and can be

regarded as generators of the algebra of area-preserving polynomial vector fields on phase space (see [9] and references therein). Note that the time-independent operators with $m = n$ are simply powers of the energy, $q_{mm} = (-H)^{m-1}$. Formally, the operator

$$\hat{q}_{mn} = \int d\lambda \hat{\psi}^\dagger(\lambda) q_{mn}(\lambda, -i\partial_\lambda) \hat{\psi}(\lambda) \quad (59)$$

implements the corresponding transformation on the fermion field theory, ignoring questions of convergence. For $m = n$ we can be more precise: Energy should be measured relative to the Fermi energy,

$$\hat{q}_{mm} = \int_{-\mu}^{\infty} d\nu (-\mu - \nu)^{m-1} b_\nu^\dagger b_\nu - \int_{-\infty}^{-\mu} d\nu (-\mu - \nu)^{m-1} b_\nu b_\nu^\dagger ; \quad (60)$$

this expression is finite for finite energy excitations away from the vacuum state with Fermi energy $-\mu$.

The operators realizing these conserved charges in the worldsheet formalism were exhibited in [43] (for recent work, see [44, 45]). The charges q_{12} and q_{21} generate the full algebra of conserved charges, so it is sufficient to write expressions for them. They are realized on the worldsheet as operators \mathcal{O}_{12} and \mathcal{O}_{21}

$$\begin{aligned} \mathcal{O}_{12} &= (\mathbf{c}\mathbf{b} + \partial\phi - \partial x)(\bar{\mathbf{c}}\bar{\mathbf{b}} + \bar{\partial}\phi - \bar{\partial}x)e^{-x-\phi} \\ \mathcal{O}_{21} &= (\mathbf{c}\mathbf{b} + \partial\phi + \partial x)(\bar{\mathbf{c}}\bar{\mathbf{b}} + \bar{\partial}\phi + \bar{\partial}x)e^{+x-\phi} . \end{aligned} \quad (61)$$

Here $\mathbf{b}(z)$, $\mathbf{c}(z)$ are the Faddeev-Popov ghosts for the local gauge choice $g_{ab} = \delta_{ab}$, *c.f.* [25, 26]. These operators have scale dimension $\Delta = \bar{\Delta} = 0$, and can be placed anywhere (unintegrated) on the two-dimensional worldsheet – moving them around changes correlators by gauge artifacts which decouple from physical quantities. The relation between matrix and continuum expressions for the conserved charges was worked out recently in [44, 45].

6 Sample calculation: the disk one-point function

An illustrative example which will allow us to compare these two rather different formulations of 2D string theory (and thereby check whether they are in fact equivalent) is the mixed correlator of one in/out state and one macroscopic loop. This correlator computes the process whereby an incoming tachyon is absorbed by the loop operator (or an outgoing one is created by the loop).

6.1 Matrix calculation

On the matrix side, we must evaluate the density-density correlator

$$\langle \text{vac} | \hat{\rho}(\lambda_1, x_1) \hat{\rho}(\lambda_2, x_2) | \text{vac} \rangle \quad (62)$$

and Laplace transform with respect to λ_1 to get the macroscopic loop, while performing LSZ reduction in λ_2 . The evaluation of (62) proceeds via substitution of (51) and use of (52) as well as the vacuum property

$$\begin{aligned} b_\nu |\text{vac}\rangle &= 0 \quad , \quad \nu > \mu \\ b_\nu^\dagger |\text{vac}\rangle &= 0 \quad , \quad \nu < \mu \end{aligned} \quad (63)$$

(note that we have not performed the usual redefinition of creation/annihilation operators below the Fermi surface). The result is [46]

$$\langle \hat{\rho}(1) \hat{\rho}(2) \rangle = \int_\mu^\infty d\nu e^{-i\nu(x_2-x_1)} \psi_\nu^\dagger(\lambda_1) \psi_\nu(\lambda_2) \int_{-\infty}^\mu d\nu' e^{i\nu'(x_2-x_1)} \psi_{\nu'}(\lambda_1) \psi_{\nu'}^\dagger(\lambda_2) . \quad (64)$$

The parabolic cylinder wavefunctions have the asymptotics (for $Y = \sqrt{\lambda^2 - 2\nu} \gg 1$, $\nu \gg 1$)

$$\psi_\nu(\lambda) \sim \left[\frac{1}{\pi Y} \right]^{1/2} \sin\left(\frac{1}{2}\lambda Y + \nu\tau(\nu, \lambda) - \frac{\pi}{4}\right) \quad (65)$$

where

$$\tau(\nu, \lambda) = - \int_{-2\sqrt{\nu}}^{-\lambda} \frac{d\lambda'}{\sqrt{\lambda'^2 - 2\nu}} = \log\left(\frac{-\lambda + \sqrt{\lambda^2 - 2\nu}}{\sqrt{2\nu}}\right) \quad (66)$$

is the WKB time-of-flight of the semiclassical fermion trajectory, as measured from the turning point of its motion.

At this point, we will make some approximations. We wish to compare the matrix and worldsheet field theory computations. However, the latter is only well-behaved in a low-energy regime, as we saw in section 3. Therefore we will approximate the energies in (64) as $\nu \sim \mu + \delta$, $\nu' \sim \mu - \delta'$, with $\delta, \delta' \ll \mu$, so that the density perturbation is very near the Fermi surface. In addition, substituting the parabolic cylinder wavefunction asymptotics (65) in (64), we drop all rapidly oscillating terms going like $\exp[\pm \frac{i}{2}\lambda^2]$; these terms should wash out of the calculation when we take $\lambda_2 \rightarrow \infty$ to perform the LSZ reduction.

With these approximations, one finds

$$\psi_\nu(\lambda_2) \psi_{\nu'}^\dagger(\lambda_2) \stackrel{\lambda_2 \rightarrow -\infty}{\sim} \frac{1}{4\pi\lambda_2} \left[\left(\sqrt{\frac{2}{\mu}} |\lambda_2| \right)^{i(\nu-\nu')} + \left(\sqrt{\frac{2}{\mu}} |\lambda_2| \right)^{-i(\nu-\nu')} \right] + O\left(\frac{\omega^2}{\mu}\right) \quad (67)$$

(recall $\omega = \nu - \nu'$). We wish to identify this with the in/out wave (56). Recall that initially the wavefunctions were multiplied by mode operators $b_{\nu'}^\dagger, b_\nu$; there is also a sum over energies. Comparing, we see that

$$\alpha_\omega = \int_0^\omega d\varepsilon b_{\omega-\varepsilon}^\dagger b_\varepsilon \times \frac{1}{2\pi} \left(\frac{\mu}{2}\right)^{-i\omega/2} \quad (68)$$

which is (up to an overall phase, which we can absorb in the definition of the operators) just the standard *bosonization formula* for 2D fermions.²¹

²¹The fermions are asymptotically relativistic in $t \pm \log |\lambda|$ as $\lambda \rightarrow -\infty$.

As for the other part of the expression, the wavefunctions at λ_1 , we make the same set of approximations, except that we use the full expression (65) rather than its $\lambda \rightarrow \infty$ limit. One finds

$$\psi_\nu^\dagger(\lambda_1)\psi_{\nu'}(\lambda_1) \sim \frac{1}{4\pi\sqrt{\lambda_1^2-2\mu}} \left[\left(\frac{-\lambda_1+\sqrt{\lambda_1^2-2\mu}}{\sqrt{2\mu}} \right)^{i(\nu-\nu')} + \left(\frac{-\lambda_1+\sqrt{\lambda_1^2-2\mu}}{\sqrt{2\mu}} \right)^{-i(\nu-\nu')} \right] + O\left(\frac{\omega^2}{\mu}\right). \quad (69)$$

Note that the terms of order ω^2/μ that have been dropped are exactly of the form to be contributions of higher topologies of worldsheet. As we saw in the scattering of waves bouncing off the exponential Liouville wall in section 3, the effective string coupling (21) is $g_{\text{eff}} \sim \omega^2/\mu$.

Fixing the sum of the energies $\nu - \nu' = \omega$ (e.g. by Fourier transformation in x), the remaining energy integral is trivial and gives a factor of ω . The macroscopic loop is finally obtained by Laplace transform with respect to λ_1 ; the answer is a Bessel function:²²

$$\int_1^\infty \left[\left(\sqrt{t^2-1} + t \right)^{i\omega} + \left(\sqrt{t^2-1} + t \right)^{-i\omega} \right] e^{-ut} \frac{dt}{\sqrt{t^2-1}} = 2 K_{i\omega}(u) \quad (70)$$

so that

$$\widetilde{W}_{i\omega}(\ell) \equiv {}_{\text{out}}\langle \text{vac} | \widetilde{W}(\ell, x) | \omega \rangle_{\text{in}} = 2 \frac{\omega}{2\pi} K_{i\omega}(\sqrt{2\mu}\ell). \quad (71)$$

The transformation (49) to z -space yields

$$\begin{aligned} W_{i\omega}(\mathbf{z}) &\equiv {}_{\text{out}}\langle \text{vac} | W(\mathbf{z}, x) | \omega \rangle_{\text{in}} = \int_0^\infty \frac{d\ell}{\ell} e^{-\ell\sqrt{2\mu}\text{ch}\pi s} \widetilde{W}_{i\omega}(\ell) \\ &= 2 \frac{\omega}{2\pi} \Gamma(i\omega) \Gamma(-i\omega) \cos(\pi s \omega) \end{aligned} \quad (72)$$

where we have parametrized $\mathbf{z} = \sqrt{2\mu} \text{ch}(\pi s)$.

The amplitude just calculated actually reveals quite a bit about the theory. We have learned that the corrections to the leading-order expressions (71), (72) are of order ω^2/μ , in agreement with the estimated higher order corrections in Liouville theory. It is a straightforward (if tedious) exercise to retain higher orders in the expansion, and thereby compute the corrections to the amplitude coming from surfaces with handles.

Another feature of Liouville theory we see appearing is its quantum wavefunction [47, 48]. In quantum theory, an operator \mathcal{O} creates a state $\mathcal{O}|0\rangle$, whose overlap with the position eigenstate $|x\rangle$ is the wavefunction $\psi_{\mathcal{O}}(x) = \langle x | \mathcal{O} | 0 \rangle$. Similarly, we wish to interpret the state created by the macroscopic loop $\widetilde{W}(\ell, x) | \text{vac} \rangle$ as the position eigenstate in the space of (ℓ, x) , whose overlap with the state $V_{i\omega} | \text{vac} \rangle$ is the wavefunction corresponding to the operator $V_{i\omega}$. This wavefunction is sometimes called the *Wheeler-de Witt* wavefunction.

²²The wavefunctions $\psi_\nu(\lambda)$ are exponentially damped under the barrier for $\nu \sim -\mu$, and so we may approximate the range of integration as $\lambda \in (-\infty, -\sqrt{2\mu})$.

Liouville correlators on surfaces with boundary developed in [13, 14, 15], building on earlier work (reviewed in [16, 17]) on closed surfaces. We will only sketch the construction; the reader interested in more details should consult these references (and the references in these references).

The basic observation is the identity

$$\partial_z^2 \mathbf{V}_{-b/2}(z) = b^2 T_{zz} \mathbf{V}_{-b/2}(z) \quad (75)$$

(and similarly for $\mathbf{V}_{-1/2b}$, *i.e.* $b \leftrightarrow 1/b$), where $\mathbf{V}_\alpha = e^{2\alpha\phi}$ are the exponential operators of Liouville field theory. This identity is consistent with the semiclassical limit $b \rightarrow 0$, $Q = b^{-1} + b \rightarrow b^{-1}$, since

$$\partial_z^2 e^{-b\phi} = [b^2 (\partial_z \phi)^2 - b \partial_z^2 \phi] e^{-b\phi} = b^2 T_{zz} e^{-b\phi} . \quad (76)$$

Correlation functions with extra insertions of T_{zz} are given in terms of those without such insertions, by the Ward identities of conformal symmetry. Thus, plugging (75) into a correlation function leads to second order differential equations on correlators involving $\mathbf{V}_{-b/2}$ (and similarly $\mathbf{V}_{-1/2b}$). Conformal invariance also dictates the structure of the correlator we wish to calculate,

$$\langle \mathbf{V}_\alpha(z) \rangle_{\mu_B} = \frac{U(\alpha)}{|z - \bar{z}|^{2\Delta_\alpha}} \quad (77)$$

where z is a coordinate on the upper half-plane, see figure 12. This is equivalent to the correlator on the disk via the conformal transformation $z = -i \frac{w+i}{w-i}$; taking into account that the operator \mathbf{V}_α transforms like a tensor of weight Δ_α in both z and \bar{z} , one finds

$$\langle \mathbf{V}_\alpha(z) \rangle_{\mu_B, \text{ disk}} = \frac{U(\alpha)}{(1 - |w|^2)^{2\Delta_\alpha}} . \quad (78)$$

The nontrivial information lies in the overall coefficient $U(\alpha)$.

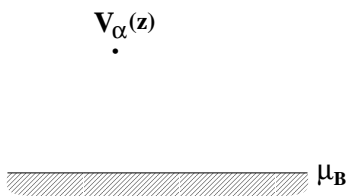


Figure 12:

In order to employ the Ward identity (75), we consider instead the two-point correlator

$$\langle \mathbf{V}_\alpha(z) \mathbf{V}_{-b/2}(w) \rangle . \quad (79)$$

The fact that $\mathbf{V}_{-b/2}$ satisfies a second order differential equation implies that only two scaling dimensions (up to integers) appear in its operator product expansion (OPE) with \mathbf{V}_α , schematically

$$\mathbf{V}_\alpha \mathbf{V}_{-b/2} \sim C_+(\alpha) [\mathbf{V}_{\alpha-b/2}] + C_-(\alpha) [\mathbf{V}_{\alpha+b/2}] , \quad (80)$$

where the square brackets denote the operator together with all that can be obtained from it by the action of the conformal algebra, the so-called *conformal block*. The differential equation coming from (75), together with (80), yields

$$\langle \mathbf{V}_\alpha(z) \mathbf{V}_{-b/2}(w) \rangle = C_+(\alpha) U(\alpha - b/2) \mathcal{G}_+(\xi) + C_-(\alpha) U(\alpha + b/2) \mathcal{G}_-(\xi) \quad (81)$$

where \mathcal{G}_\pm are hypergeometric functions of the cross-ratio $\xi = \frac{(z-w)(\bar{z}-\bar{w})}{(z-\bar{w})(\bar{z}-w)}$. What are the coefficients $C_\pm(\alpha)$? For $C_+(\alpha)$ the result of the OPE satisfies conservation of the “charge” of the exponential (the Liouville zero-mode momentum p_ϕ). Even though this momentum is not conserved due to the presence of the tachyon wall, which violates translation invariance in ϕ , if we nevertheless use free field theory to evaluate it we find trivially $C_+(\alpha) = 1$. We similarly use naive perturbation theory in powers of μ to evaluate $C_-(\alpha)$, bringing down the tachyon potential $\int \mu e^{2b\phi}$ in a power series expansion and evaluating the resulting integrated correlation functions using free field theory. Only the first term in the μ expansion contributes, and we find

$$\begin{aligned} C_-(\alpha) &= \left\langle \mathbf{V}_\alpha(0) \mathbf{V}_{-b/2}(1) \mathbf{V}_{Q-\alpha-b/2}(\infty) \left(-\mu \int d^2z \mathbf{V}_b(z, \bar{z}) \right) \right\rangle_{\text{FFT}} \\ &= -\pi\mu \frac{\gamma(2b\alpha - 1 - b^2)}{\gamma(-b^2)\gamma(2b\alpha)} \end{aligned} \quad (82)$$

where $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$.²⁴

Why are we allowed to use a perturbative expansion in μ and free field theory for evaluating these quantities? After all, the loop amplitude $\mu^{-i\omega/2} K_{i\omega}(\sqrt{2\mu}\ell)$ is certainly *not* polynomial in μ . Nevertheless, for special “resonant” amplitudes this procedure is justified. Resonant amplitudes are those for which the sum of the exponents $\sum \alpha_i$ of the collection of Liouville operators \mathbf{V}_{α_i} adds up to a negative multiple of the exponent $2b$ of the Liouville potential. In such cases, the path integral can be evaluated by perturbation theory in μ . This feature is related to the property that the integral over the constant mode of ϕ in the path integral is dominated by the region $\phi \rightarrow -\infty$, where the Liouville potential is effectively vanishing. The use of free field theory methods is then justified. The correlators that define C_\pm satisfy this resonance condition. Note that we are not using free field theory to evaluate the full amplitude, but rather only to evaluate the operator product coefficients with the special degenerate operator $\mathbf{V}_{-b/2}$ (and similarly $\mathbf{V}_{-1/2b}$).

²⁴ $\mathbf{V}_{Q-\alpha-b/2}$ is the conjugate operator to $\mathbf{V}_{\alpha+b/2}$ in free field theory with a linear dilaton of slope Q .

We now have partial information on the correlation function. To get a closed system of equations, we need a second relation on (79). For this purpose, we consider the OPE of $\mathbf{V}_{-b/2}(w)$ with its image across the boundary to make the identity operator, by taking $w \rightarrow \bar{w}$ (in the process, we need to transform to another basis for the hypergeometric functions \mathcal{G}_{\pm} adapted to this particular degeneration). In this limit, the correlator (79) factorizes,

$$\langle \mathbf{V}_{\alpha}(z) \mathbf{V}_{-b/2}(w) \rangle_{\mu_B} \stackrel{w \rightarrow \bar{w}}{\sim} \langle \mathbf{V}_{\alpha}(z) \rangle_{\mu_B} \langle \mathbf{V}_{-b/2}(w) \rangle_{\mu_B} . \quad (83)$$

The first factor on the right-hand side is given simply in terms of $U(\alpha)$, and the second factor is yet another resonant amplitude, which we can evaluate in free field theory by bringing down the boundary cosmological constant interaction from the action:

$$(\text{Im } w)^{2\Delta_{\alpha}} \left\langle \mathbf{V}_{-b/2}(w) B_Q(\infty) \left(-\mu_B \oint d\xi B_b(\xi) \right) \right\rangle_{\text{FFT}} = -2\pi\mu_B \frac{\Gamma(1-2b^2)}{\Gamma^2(-b^2)} . \quad (84)$$

Here the integral over ξ is along the boundary, which is the real axis; B_{α} is the operator $e^{2\alpha\phi}$ inserted on the boundary; and B_Q represents the extrinsic curvature of the boundary at infinity.

Equating the two expressions (81) and (83), and using (82), (84), one arrives at a shift relation on $U(\alpha)$ [13, 14, 15],

$$-\frac{2\pi\mu_B}{\Gamma(-b^2)} U(\alpha) = \frac{\Gamma(-b^2+2b\alpha)}{\Gamma(-1-2b^2+2b\alpha)} U(\alpha-b/2) - \frac{\pi\mu\Gamma(-1-b^2+2b\alpha)}{\gamma(-b^2)\Gamma(2b\alpha)} U(\alpha+b/2) . \quad (85)$$

There is a similar shift relation obtained by use of $\mathbf{V}_{-1/2b}$. It is convenient to write μ_B in terms of a parameter s via

$$\cosh^2(\pi bs) = \frac{\mu_B^2}{\mu} \sin(\pi b^2) ; \quad (86)$$

then the two discrete shift relations (obtained by use of both $\mathbf{V}_{-b/2}$ and $\mathbf{V}_{-1/2b}$) are solved by

$$U(\alpha) = \frac{2}{b} \left(\pi\mu\gamma(b^2) \right)^{\frac{Q-2\alpha}{2b}} \Gamma(2b\alpha-b^2) \Gamma\left(\frac{2\alpha}{b} - \frac{1}{b^2} - 1\right) \cosh[(2\alpha-Q)\pi s] . \quad (87)$$

For the vertex operators with $\alpha = \frac{1}{2}Q + \frac{i}{2}\omega$ appearing in the scattering amplitudes, this translates into

$$U(\alpha = \frac{1}{2}Q + \frac{i}{2}\omega) = 2i\omega \left(\pi\mu\gamma(b^2) \right)^{-i\omega/2b} \Gamma(ib\omega) \Gamma(i\omega/b) \cos(\pi s\omega) . \quad (88)$$

The shift operator relations don't fix the overall normalization of $U(\alpha)$. This normalization is obtained by demanding that the residues of the poles at $2\alpha = Q - nb$

(i.e., $i\omega = -nb$) for $n = 1, 2, 3, \dots$, agree with the “resonant amplitude” integrals for these special momenta.

Note that the full set of resonant amplitude integrals involve bringing down powers of both $\mu e^{2b\phi}$ and also $\tilde{\mu} e^{(2/b)\phi}$ from the action. One needs to use the complete set in order to provide sufficient constraints to fully determine the Liouville correlators. Hence *both* are present in the theory; moreover, one finds for consistency that their coefficients must be related:

$$\pi \tilde{\mu} \gamma(1/b^2) = [\pi \mu \gamma(b^2)]^{1/b^2} . \quad (89)$$

It turns out that this is more or less the relation implied by the analytic continuation of the amplitude for reflection off the Liouville potential

$$\tilde{\mu}/b = \mu b \mathcal{R}(\omega = i(Q - 2b)) . \quad (90)$$

The reflection amplitude $\mathcal{R}(\omega)$ for $V_{i\omega} \rightarrow V_{-i\omega}$ may be read off the two-point correlation function for tachyon vertex operators. A similar relation holds for the boundary cosmological constant; the boundary interaction is actually

$$\delta S S_{\text{bdy}} = \oint \left(\mu_B e^{b\phi} + \tilde{\mu}_B e^{(1/b)\phi} \right) \quad (91)$$

with

$$\cosh^2(\pi s/b) = \frac{\tilde{\mu}_B^2}{\tilde{\mu}} \sin(\pi/b^2) . \quad (92)$$

Thus there is a kind of strong/weak coupling duality in Liouville QFT, characterized by

$$b \leftrightarrow 1/b \quad , \quad \mu \leftrightarrow \tilde{\mu} \quad , \quad \mu_B \leftrightarrow \tilde{\mu}_B \quad (93)$$

(recall that $b \rightarrow 0$ was the weak coupling limit of Liouville theory). The parameter s is invariant under this transformation.

6.3 Comparing the results

Finally, we are ready to compare the two approaches. First we must assemble the Liouville disk amplitude with the contributions of the free matter field X and the Faddeev-Popov ghosts. There is a factor of $1/2\pi$ from gauge fixing the conformal isometries of the punctured disk (rotations around the puncture). The disk expectation value of the matter is

$$\left\langle e^{i\omega X}(z) \right\rangle_{\text{Dirichlet}} = \frac{1}{|z - \bar{z}|^{2\Delta_\omega}} \quad (94)$$

(equivalently, $(1 - |w|^2)^{-2\Delta_\omega}$ if we are working on the disk rather than the upper half-plane), which simply reflects the fact that the Dirichlet boundary condition on X is a delta function (and hence its Fourier transform is one). The factors of $|z - \bar{z}|$

cancel among Liouville, matter, and ghosts (we must take $b \rightarrow 1$ in the Liouville part since $D = 2$; this involves a multiplicative renormalization of μ and μ_B in order to obtain finite results). This cancellation of coordinate dependence merely reflects that we have correctly calculated a conformally invariant and therefore physical amplitude. Thus

$$\langle V_{i\omega}(z, \bar{z}) \rangle_{\text{disk}} = \frac{1}{2\pi} 2i\omega \hat{\mu}^{-i\omega/2} (\Gamma(i\omega))^2 \cos(\pi s\omega) \quad (95)$$

where we have defined

$$\hat{\mu} = \pi\mu\gamma(b^2) \stackrel{b \rightarrow 1}{\sim} 2\pi\mu(1-b) \quad (96)$$

as the quantity to be held fixed in the $b \rightarrow 1$ limit.

Comparing to the matrix model result (72), we find the same result provided that we identify

$$\begin{aligned} V_{i\omega}^{\text{matrix}} &= (\hat{\mu})^{i\omega/2} \frac{\Gamma(-i\omega)}{\Gamma(i\omega)} V_{i\omega}^{\text{continuum}} \\ \frac{1}{2}\mu^{\text{matrix}} &= \hat{\mu}^{\text{continuum}} \\ \frac{1}{2}\mu_B^{\text{matrix}} &= \frac{1}{2}\mathcal{Z}_{\text{matrix}} = \hat{\mu}_B^{\text{continuum}} \equiv 2\pi\mu_B^{\text{cont}}(1-b) . \end{aligned} \quad (97)$$

(the last relation amounts to $\hat{\mu}_B = \sqrt{\hat{\mu}} \text{ch}(\pi s)$). Thus the exact evaluation of the worldsheet amplitude allows a precise mapping between the continuum and matrix approaches.

The energy-dependent phase in the relative normalizations of $V_{i\omega}$ results in a varying time delay of reflection for particles of different energy. It was shown in [49] that this time delay reproduces what one would expect based on the gravitational redshift seen by one particle after another has been sent in. Thus the so-called “leg-pole factor” $\frac{\Gamma(-i\omega)}{\Gamma(i\omega)}$ in equation (97) is an important physical effect, which is added by hand to the matrix model. It is not yet understood if there is a derivation of this factor from first principles in the matrix model.

Other amplitudes that have been computed on both sides of the correspondence and shown to agree include

- The tree level S-matrix [50, 51, 42],²⁵
- The torus partition function [52, 53],
- The disk one-point function calculated above [46, 47, 48, 13, 14],
- The annulus correlation function for two macroscopic loops [47, 19].

²⁵The leg-pole factor (97) in the relative normalization of vertex operators was first observed here, and shown to be a property of all the tree amplitudes.

One can also show that the properties of the ground ring of conserved charges defined in section 5.3 agree between the matrix and continuum formulations, at leading order in $1/\mu$ [22]. For instance, on the sphere one calculates using the Liouville OPE coefficients C_{\pm} that

$$\langle \mathcal{O}_{12} \mathcal{O}_{21} \rangle = \langle \mathcal{O}_{22} \rangle = \langle -H \rangle = \mu . \quad (98)$$

This result is consistent with the fact that perturbative excitations live at the Fermi surface, where the energy is $H = \frac{1}{2}(p_{\lambda}^2 - \lambda^2) = -\mu$.

Thus the matrix approach is reproducing the quantum dynamics of Liouville CFT coupled to a free field. Note that the matrix approach is much more economical computationally, and we immediately see how to compute the higher order corrections (just go to higher order in $1/\mu$ in our approximations); for Liouville, we need to work *much* harder – we need to go back to the conformal bootstrap and compute correlation functions on the disk with handles, then integrate over the moduli space.

7 Worldsheet description of matrix eigenvalues

Finally, what about the eigenvalues themselves? They are gauge invariant observables which are manifest in the matrix formulation; what is their description in the continuum formalism? Note that this question bears on the continuum description of nonperturbative phenomena such as the eigenvalue tunnelling which leads to the nonperturbative instability of the model. Experience from string theory in higher dimensions (*e.g.* black hole microphysics) has taught us that D-brane dynamics provides a description of strong coupling physics. Therefore we should examine the D-branes of 2D string theory. The fact that the tension of D-branes is naively $O(1/g_s)$ means that they are the natural light degrees of freedom in the strong coupling region.

In the worldsheet description of dynamics, a D-brane is an object which puts boundaries on the worldsheet. The boundary conditions on the worldsheet fields X^{μ} tell us about the position of the brane and the boundary interactions in the worldsheet action specify the background fields localized on the brane. Perturbations of the boundary background fields are (marginal) scaling operators on the boundary. The theory thus has two sectors of strings – *open strings* that couple to worldsheet boundaries (D-branes), and *closed strings* that couple to the bulk of the worldsheet.

In a sense, the macroscopic loop is a spacelike D-brane – one with Dirichlet boundary conditions in the timelike direction X and Neumann boundary conditions in the spacelike direction ϕ . The boundary interaction $\mu_B \oint e^{b\phi}$ is a “boundary tachyon” that keeps ϕ_{bdy} away from the strong coupling region $\phi \rightarrow \infty$ (at least for the appropriate sign of μ_B). This D-brane is however a collective observable at fixed time x of the matrix model, and not a dynamical object. A depiction of the D-brane interpretation of the calculation of section 6 is shown in figure 13a.

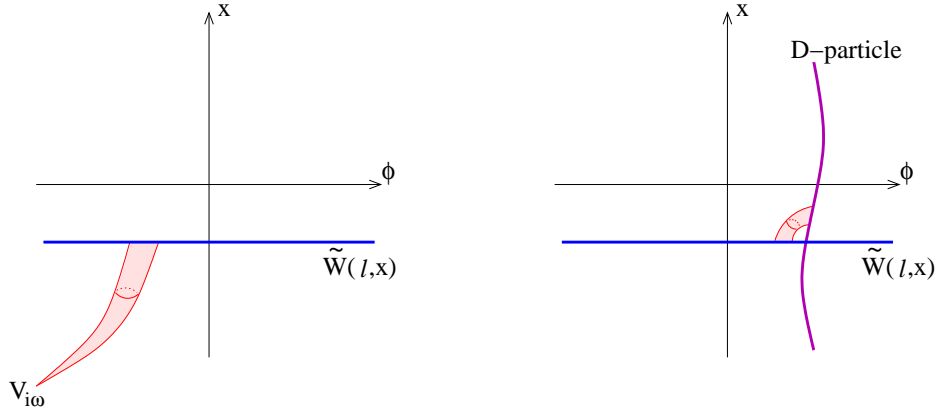


Figure 13: (a) A macroscopic loop is a spatial D-brane that absorbs and emits closed strings. (b) The loop is also a probe of the motion of D-particles.

Instead, the matrix eigenvalue is localized in the spatial coordinate λ and hence quasi-localized in ϕ . Here it is important to recall that ϕ and λ are related by the integral transform (49), and are thus not directly identified. Nevertheless, localized disturbances in ϕ bouncing off the exponential Liouville wall are related to localized disturbances of the Fermi surface in λ bouncing off the inverted oscillator barrier, so there is a rough equivalence.

Therefore, we consider Dirichlet boundary conditions for ϕ . Since ϕ shifts under local scale transformations ($e^{2b\phi}g_{ab}$ is the dynamical metric), the Dirichlet boundary condition

$$\phi|_{\text{bdy}} = \phi_0 \quad (99)$$

is not conformally invariant unless $\phi = \pm\infty$. Now $\phi = -\infty$ is the weak coupling asymptotic boundary of ϕ space, and corresponds to boundaries of zero size, which we usually think of as punctures in the worldsheet where local vertex operators are inserted. On the other hand, $\phi = +\infty$ is what we want, a boundary deep inside the Liouville wall at strong coupling.

In fact, we know a classical (constant negative curvature) geometry with this property:

$$ds^2 = e^{2b\phi} dz d\bar{z} = \frac{Q}{\pi\mu b} \frac{dz d\bar{z}}{(1 - z\bar{z})^2}, \quad (100)$$

the Poincaré disk (or Lobachevsky plane). Proper distances blow up toward the boundary: $\phi \rightarrow \infty$, as advertised.

For this D-brane to move in time, the boundary condition in X should be Neumann. What sort of conformally invariant boundary interaction can we have? Since ϕ is fixed on the boundary, the interaction can only involve X ; conformal invariance

then dictates

$$\begin{aligned}\delta SS_{\text{bdy}} &= \beta \oint \cos(X) \quad , \quad (X \text{ Euclidean}) \\ \delta SS_{\text{bdy}} &= \begin{cases} \beta \oint \cosh(X) \\ \beta \oint \sinh(X) \end{cases} \quad , \quad (X \text{ Lorentzian})\end{aligned}\tag{101}$$

This interaction is the boundary, open string analogue of the closed string tachyon background $V(X)$ in (5); it describes an open string ‘tachyonic mode’ of the D-brane, since the interaction grows exponentially in Lorentz signature spacetime.²⁶

The open string tachyon (101) describes the decay of an unstable D-particle located in the strong coupling region $\phi \rightarrow \infty$. The tachyon condensate in Lorentz signature looks promising to be the description of an eigenvalue in the matrix model, whose classical motion is

$$\begin{aligned}\lambda(x) &= \lambda_0 \cosh(x) \quad , \quad E = -\frac{1}{2}\lambda_0^2 < 0 \\ \lambda(x) &= \lambda_0 \sinh(x) \quad , \quad E = +\frac{1}{2}\lambda_0^2 > 0\end{aligned}\tag{102}$$

depending on whether the eigenvalue passes over, or is reflected by, the harmonic barrier. Similarly, the Euclidean trajectory $\lambda(x) = \lambda_0 \cos(x)$ is oscillatory, appropriate to the computation of the WKB tunnelling of eigenvalues under the barrier.

How do we see that this is so? In [22] (building on earlier work [18, 20]) this result was demonstrated by computing the ground ring charges \mathcal{O}_{12} and \mathcal{O}_{21} on the disk, and showing that they give the classical motions above. Here we will employ a complementary method: We will probe the D-brane motion with the macroscopic loop. This will exhibit the classical motion quite nicely.

7.1 Lassoing the D-particle

The matrix model calculation of a macroscopic loop probing a matrix eigenvalue is trivial. Recall that the macroscopic loop is

$$W(\mathbf{z}, x_0) = -\frac{1}{N} \text{tr} \log(\mathbf{z} - M(x_0)) = -\int d\lambda \hat{\rho}(\lambda, x_0) \log(\mathbf{z} - \lambda) .\tag{103}$$

An individual eigenvalue undergoing classical motion along the trajectory $\lambda(x_0)$ gives a delta-function contribution to the eigenvalue density

$$\delta\rho(\lambda, x_0) = \delta(\lambda - \lambda(x_0))\tag{104}$$

where $\lambda(x_0) = -\lambda_0 \cos(x_0)$ for Euclidean signature, and $\lambda(x_0) = -\lambda_0 \cosh(x_0)$ for Lorentzian signature. Plugging into (103), we find

$$W_{\text{eval}}(\mathbf{z}, x_0) = -\log[\mathbf{z} - \lambda(x_0)] .\tag{105}$$

²⁶The mass shell condition for boundary interactions describing background fields on the D-brane can be computed along the lines of (13)-(17), except that the OPE of the stress tensor with a boundary operator should be performed using the appropriate Dirichlet or Neumann propagator, $X(z)X(w) \sim -\frac{\alpha'}{2}(\log|z-w|^2 \mp \log|z-\bar{w}|^2)$.

In the worldsheet formalism, the presence of a macroscopic loop introduces a second boundary, besides the one describing the D-particle. The leading order connected correlator of the loop and the D-particle is thus an annulus amplitude; the worldsheet and boundary conditions are depicted in figure 14, while the spacetime interpretation is shown in figure 13b. The parameter τ is an example of a modulus of the surface, the Schwinger parameter for the propagation of a closed string, which cannot be gauged away by either reparametrizations or local scale transformations; in the end, we will have to integrate over it.

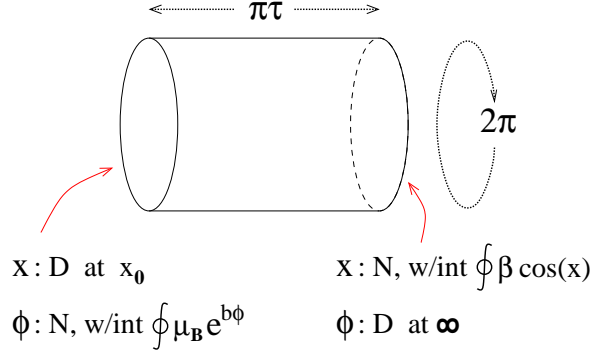


Figure 14: *The annulus worldsheet describing a macroscopic loop probing a D-particle.*

There are two ways to think about this worldsheet as the propagation of a string. If we view worldsheet time as running around the circumference of the annulus, we think of the diagram as the one-loop vacuum amplitude of an open string, a string having endpoints. At one endpoint of the string, we classically have the boundary condition

$$\partial_n \phi = 2\pi\mu_B e^{b\phi} \quad , \quad X = x_0 \quad (106)$$

describing the macroscopic loop; at the other end, we have

$$\phi = \infty \quad , \quad \partial_n X = 2\pi\beta \sin(X) \quad (107)$$

describing the moving D-particle. On the other hand, we can think of the diagram as the propagation of a closed string for a worldsheet time $\pi\tau$, folded into “boundary states” $|B\rangle$ which implement the boundary conditions on the fields. These boundary states are completely determined by these conditions, *e.g.*

$$\begin{aligned} (\partial_n \phi - 2\pi\mu_B e^{b\phi})|B_N(\mu_B)\rangle_\phi &= 0 \\ (X - x_0)|B_D(x_0)\rangle_X &= 0 \end{aligned} \quad (108)$$

and so on. Because the Liouville and matter fields do not interact, the boundary state factorizes into the tensor product of the boundary states for X and for ϕ . The Liouville partition function can then be written

$$\begin{aligned}\mathcal{Z}_L(q) &= \langle B_N, \mu_B | e^{-\pi\tau H} | B_D \rangle \\ &= \int d\nu \Psi_{\text{FZZT}}^*(\nu, \mu_B) \Psi_{\text{ZZ}}(\nu) \frac{q^{\nu^2}}{\eta(q)}\end{aligned}\quad (109)$$

where $q = \exp[-2\pi\tau]$, and the Dedekind eta function $\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ represents the contribution to the partition function of all the Liouville oscillator modes. The quantities Ψ_{FZZT} and Ψ_{ZZ} are the zero mode parts of the Neumann and Dirichlet boundary state wavefunctions, respectively; see [13, 14] and [15], respectively. Explicitly,

$$\begin{aligned}\Psi_{\text{FZZT}}(\nu, \mu_B) &= \cos(2\pi\nu s) \left[\frac{\Gamma(1 + 2i\nu b) \Gamma(1 + 2i\nu/b)}{2^{1/4} (-2\pi i\nu)} \hat{\mu}^{-i\nu/b} \right] \\ \Psi_{\text{ZZ}}(\nu) &= 2 \sinh(2\pi\nu/b) \sinh(2\pi\nu b) \left[\frac{\Gamma(1 + 2i\nu b) \Gamma(1 + 2i\nu/b)}{2^{1/4} (-2\pi i\nu)} \hat{\mu}^{-i\nu/b} \right]\end{aligned}\quad (110)$$

Here, s parametrizes μ_B as in equation (86). The “Neumann” wavefunction Ψ_{FZZT} is the one obtained before, from the macroscopic loop calculation; ν is the Liouville zero-mode momentum $\alpha = \frac{1}{2}Q + i\nu$ in the “closed string channel”. This is not surprising; before we used the macroscopic loop to probe the wavefunction of a scattering state, now we are using it to probe a D-brane state to see if it has the properties of a matrix eigenvalue.

The authors of [15] showed that $\Psi_{\text{ZZ}}(\nu)$ has the property that all operators behave like the identity operator as they approach the corresponding boundary (so that one approaches the constant negative curvature “vacuum” near the boundary of the Poincaré disk). Ordinarily in Liouville theory, when an operator such as \mathbf{V}_α approaches the boundary $z = \bar{z}$ (*e.g.* with boundary condition (106)), it expands as a sum of boundary operators B_β . For the boundary state with wavefunction $\Psi_{\text{ZZ}}(\nu)$, only the identity boundary operator $B_0 = \mathbb{1}$ appears in the limit $z \rightarrow \bar{z}$.

Now for the matter partition function. The annulus partition function with the requisite boundary conditions was computed in [54] for Euclidean X , with the result

$$\mathcal{Z}_X = \frac{1}{\sqrt{2}\eta(q)} \sum_{n=-\infty}^{\infty} q^{n^2/4} \cos[n\pi(\frac{1}{2} + \gamma)] \quad , \quad \sin(\pi\gamma) \equiv \cos(x_0) \sin(\pi\beta) \quad . \quad (111)$$

Again the Dedekind eta function represents the contribution of the X oscillator modes, and the sum results from the zero modes. The Faddeev-Popov ghost partition function is

$$\mathcal{Z}_{\text{gh}} = \eta^2(q) \quad , \quad (112)$$

cancelling the oscillator η functions of ϕ and X . This is related to the fact that there are no transverse directions in which the string can oscillate – at generic momenta just the tachyon, with only center of mass motion of the string, is physical.

Combining all the contributions, we have

$$\begin{aligned}\mathcal{Z} &= \int_0^\infty d\tau \mathcal{Z}_L \cdot \mathcal{Z}_X \cdot \mathcal{Z}_{\text{gh}} \\ &= \int_0^\infty d\tau \int d\nu \cos(2\pi\nu s) \sum_{n=-\infty}^\infty \cos[n\pi(\tfrac{1}{2} + \gamma)] q^{\nu^2 + n^2/4} .\end{aligned}\quad (113)$$

Doing the τ integral, and the ν integral by residues,²⁷ one finds

$$\mathcal{Z} = 2 \sum_{n=1}^\infty \frac{1}{n} \exp[-n\pi s] \cos[n\pi(\tfrac{1}{2} + \gamma)] . \quad (114)$$

The sum is readily performed, and after a little algebra, one obtains

$$\mathcal{Z} = -\log[2(\cosh(\pi s) + \sin(\pi\gamma))] . \quad (115)$$

Define now

$$\lambda(x_0) = -\sqrt{2\mu} \sin(\pi\gamma) = -\sqrt{2\mu} \sin(\pi\beta) \cos(x_0) \equiv -\lambda_0 \cos(x_0) \quad (116)$$

and recall that $\mu_B = \sqrt{2\mu} \text{ch}(\pi s) = \mathbf{z}$; then we have

$$\mathcal{Z} = -\log[\mathbf{z} - \lambda(x_0)] + \tfrac{1}{2} \log(\mu/2) . \quad (117)$$

The additive constant is ambiguous, and depends on how we regularize the divergent term in (113); nevertheless, it is independent of the boundary data for X and ϕ , and so does not affect the measurement of the D-particle motion.²⁸ Dropping this last term, we finally reproduce the result (so easily found) for the probe of eigenvalue motion in the matrix model, equation (105)!

Note that, even though the Dirichlet boundary condition on ϕ is in the strong coupling region $\phi \rightarrow \infty$, the wavefunctions are such that we obtain sensible results for the amplitude. Note also that the boundary interaction for the D-particle depends only on X , and thus the D-particle naively is not moving in ϕ . This is a cautionary tale, whose moral is to compute physical observables! Nevertheless, when the D-particle reaches the asymptotic region of weak coupling, it should be moving in both ϕ and λ . Somehow the field space coordinate of the open string tachyon and the ϕ coordinate of spacetime become related in the course of the tachyon's condensation, and it remains to be understood how this occurs.

The consideration of multiple D-particles elicits the matrix nature of their open string dynamics. Now we must add to the description of the boundary state a finite dimensional (*Chan-Paton*) Hilbert space \mathcal{H}_{CP} describing which D-particle a given

²⁷The $n = 0$ term is divergent and needs to be regularized. Our choice is to replace the double pole at $\nu = 0$ in this term by $\frac{1}{\nu^2 + \epsilon^2}$, and then subtract the pole term in ϵ after the ν integral is evaluated by residues.

²⁸The extra term $(\frac{1}{\epsilon} - \frac{1}{2} \log(\mu_B))$ can be thought of as the regularized volume of ϕ -space [55].

worldsheet boundary is attached to. Open string operators act as operators on this finite-dimensional Hilbert space, *i.e.* they are matrix-valued (equivalently, an open string is an element of $\mathcal{H}_{\text{CP}} \otimes \mathcal{H}_{\text{CP}}^*$ specifying the Chan-Paton boundary conditions at each end).

The open string tachyon is now a matrix field; the parameter β in equation (101) is a matrix of couplings β_{ij} , $i, j = 1, \dots, n$ for n D-particles. There is an additional possible boundary interaction

$$\delta S S_{\text{bdy}} = \oint A_{ij} \partial_t X \quad (118)$$

which is a matrix gauge field on the collection of D-particles. We ignored it in our previous discussion because its role is to implement Gauss' law on the collection of D-particles, which is trivial in the case of a single D-particle. When several D-particles are present, however, this Gauss law amounts to a projection onto $U(n)$ singlet states. Thus the continuum description suggests that the $U(N)$ symmetry of the matrix mechanics is gauged, which as mentioned in section 4.1 projects the theory onto $U(N)$ singlet wavefunctions. Singlet sector matrix mechanics looks very much like the quantum mechanics of N D-particles in 2D string theory.

Several ingredients of the relation between the continuum and matrix formulations remain to be understood. The probe calculation tells us that the open string tachyon condensate on the D-particle describes its leading order, classical trajectory. One should understand how higher order corrections lead to the quantum corrections for the wavefunction of a quantum D-particle, and show that this series matches the WKB series for the wavefunctions of the eigenvalue fermions of the matrix model. Also, the description in the continuum formulation of an eigenvalue as a D-particle is quite different from the ensemble of eigenvalues in the Fermi sea, whose collective dynamics is expressed via continuum worldsheets. Under what circumstances is eigenvalue dynamics that of D-particles, as opposed to that of closed string worldsheets? For instance, the $U(n)$ symmetry of a collection of n D-particles should extend to the full $U(N)$ symmetry of the whole matrix. How do we see this larger symmetry in the continuum formulation? We cannot simply turn all the fermions into D-particles; there would then be nothing left to make the continuum worldsheets that attach to these branes. The continuum formalism is really adapted to describing a small number of matrix eigenvalues that have been separated from rest of the ensemble, thus leading to distinct treatment of the few separated ones as D-particles, and the vast ensemble of remaining ones as the threads from which continuum worldsheets are woven.

7.2 Summary

To summarize, we have an expanding translation table between matrix and continuum formalisms:

Continuum	Matrix
Closed string vacuum	Fermi sea of matrix eigenvalues
Liouville potential $\mu e^{2b\phi}$	Inverted oscillator barrier
Worldsheet cosmological constant μ	Fermi energy -2μ
Strings	Fermi surface density wave quanta
String S-matrix	Density wave S-matrix
D-branes w/ X : D, ϕ : N, $SS_{\text{bdy}} = \mu_B \oint e^{b\phi}$	Macroscopic loops $\text{tr}[\log(z - M(x))]$
Boundary cosm. const. μ_B	Loop eigenvalue parameter $2z$
D-branes w/ ϕ : D, X : N, $SS_{\text{bdy}} = \beta \oint \cos X$	Eigenvalues outside the Fermi sea
Open string tachyon coupling β	Eigenvalue energy $E = -\mu \sin^2 \pi \beta$
Open string tachyon on n D-particles	A block of the matrix M

A similar dictionary is known for the fermionic string, which will be described briefly in the next section. Here one has the added advantage that the model is nonperturbatively well-defined. In these models both sides of the duality are again calculable. One may hope that open/closed string duality can be worked out in complete detail in this example, and that it will lead to valuable insights into the general class of open/closed string dualities to which it belongs.

8 Further results

8.1 Fermionic strings

The remarkable agreement between the continuum and matrix formulations of 2D string theory leads us to believe that they are equivalent. However, in the case of the bosonic string, both are asymptotic expansions. Worldsheet perturbation theory is an asymptotic expansion, and it was our hope that, as in higher dimensional gauge/gravity correspondences, the matrix (gauge) theory formulation would provide a nonperturbative definition of the theory. But the nonperturbative instability of the vacuum to eigenvalue tunnelling across to the right-hand side of the oscillator barrier means that the theory does not really exist after all.

An obvious fix for this difficulty would be to fill up the other side of the barrier with fermions as well (see *e.g.* [56] for an example of this proposal). But this leads to an equally obvious question: We found agreement with continuum bosonic strings using just the fluctuations on one side of the oscillator barrier. What do the fluctuations on the other side describe? Perturbatively, they are a second, decoupled copy of the same dynamics. Nonperturbatively, the two sides of the barrier communicate, by tunnelling and by high energy processes that pass over the barrier. It is now understood [21, 22] that this stable version of the matrix model describes 2D fermionic string theory (the type 0B string, in the arcane terminology of the subject).

The fermionic string extends the construction of section 2 by supersymmetrizing the worldsheet theory: The spacetime coordinates $X^\mu(\sigma)$ of the worldsheet path integral gain superpartners $\psi^\mu(\sigma)$ which transform worldsheet spinors (and spacetime vectors). The local reparametrization and scale invariance condition generalizes to local supersymmetry and super-scale invariance; in other words, the stress tensor T has a superpartner G , and the dynamical condition is that both must vanish in correlation functions.

If we perform the same exercise in the path integral formulation (1) of the particle propagator in flat spacetime, the quantization of the superpartner ψ leads to equal time anticommutation relations

$$\{\psi^\mu, \psi^\nu\} = \delta^{\mu\nu} . \quad (119)$$

One realization of these anticommutation relations is to represent the ψ 's as Dirac matrices. The quantum mechanical Hilbert space contains not only the position wavefunction, but also a finite dimensional spin space in which the ψ^μ act – the particle being propagated is a spinor. Thus worldsheet supersymmetry is a way to introduce spacetime fermions into a worldline or worldsheet formalism.²⁹ A second realization of the anticommutation relations (119), using complex fermions, treats ψ_μ^* as a creation operator, and its conjugate ψ^μ as an annihilation operator. Starting from the fermion ‘vacuum’ $|0\rangle$, $\psi^\mu|0\rangle = 0$, the set of polarization states propagated along the particle worldline, $\{\psi_{\mu_1}^* \cdots \psi_{\mu_r}^*|0\rangle\}$, transform as a collection of antisymmetric tensors $C_{\mu_1 \dots \mu_r}$ in spacetime.

The same story arises in the string generalization; the worldsheet fermions ψ^μ can realize a collection of antisymmetric tensors in spacetime, or under suitable conditions the propagating strings are spinors in spacetime. The so-called type 0 fermionic strings do not realize spacetime fermions, but do contain the antisymmetric tensor fields. We can divide the set of antisymmetric tensor fields into those with even rank and those with odd rank. The type 0A theory involves a projection onto odd rank tensors (with even rank field strength), while the type 0B theory contains even rank tensors (with odd rank field strength). In particular, the type 0B theory contains a 0-form or scalar potential C in addition to the tachyon V . This scalar provides the needed extra degrees of freedom to represent, in the worldsheet formalism, the density oscillations on either side of the harmonic barrier in the matrix model with both sides filled.

To describe the vertex operator for this scalar requires a bit of technology [57]. One can think of the left- and right-moving worldsheet fermions $\psi(z)$, $\bar{\psi}(\bar{z})$ in terms of the 2d Ising model. In addition to the fermion operators, the Ising model has order and disorder operators $\sigma(z, \bar{z})$ and $\mu(z, \bar{z})$, often called *spin fields*. In 2D string theory, we thus have the spacetime coordinate fields X , ϕ and their superpartners ψ_X , ψ_ϕ , as well as the spin fields σ_X , σ_ϕ , μ_X , μ_ϕ , as the ingredients out of which we

²⁹Note that the wave operator is the Hamiltonian for the quantum mechanics, *e.g.* $H = -p^2 + m^2$ in flat spacetime; the supersymmetry algebra $Q^2 = H$ points to the fact that the supercharge $Q = p \cdot \psi$ is the Dirac operator in this context.

can build vertex operators (there may also be contributions from Faddeev-Popov ghosts if this is required by gauge invariance). The on-shell tachyon vertex is now

$$V_{i\omega} = (\psi_X \pm \psi_\phi)(\bar{\psi}_X \pm \bar{\psi}_\phi) e^{i\omega(X \pm \phi)} e^{Q\phi} , \quad (120)$$

and there is also the second (so-called RR) scalar, with vertex operator

$$C_{i\omega} = \Sigma_{\text{gh}}(\sigma_X \sigma_\phi \pm \mu_X \mu_\phi) e^{i\omega(X \pm \phi)} e^{Q\phi} . \quad (121)$$

Here, Σ_{gh} is a spin field for the Faddeev-Popov ghosts arising from fixing local supersymmetry [57]. It was shown in [51] that the tree-level S-matrix amplitudes for the linear combinations

$$T_{L,R}(\omega) = \frac{\Gamma(-i\omega\sqrt{\alpha'/2})}{\Gamma(i\omega\sqrt{\alpha'/2})} V_{i\omega} \pm \frac{\Gamma(\frac{1}{2}-i\omega\sqrt{\alpha'/2})}{\Gamma(\frac{1}{2}+i\omega\sqrt{\alpha'/2})} C_{i\omega} \quad (122)$$

decouple from one another, *i.e.* the connected amplitudes involving both sets of operators $T_{L,R}$ vanish; and the amplitudes involving just one set are the same as for the bosonic string, up to a rescaling $\alpha' \rightarrow 2\alpha'$. This strongly suggests we identify $T_{L,R}$ as the asymptotic modes of density fluctuations on the left and right sides of the harmonic barrier in the symmetrically filled matrix model. Note that there are again energy-dependent phases involved in the relation between matrix model asymptotic states and continuum asymptotic states. One should think of the fields V and C as corresponding to the symmetric and antisymmetric perturbations of the Fermi sea of the matrix model, after these phases are stripped off. This identification is consistent with the fact that S-matrix amplitudes vanish for an odd number of parity-odd density perturbations; the \mathbb{Z}_2 Ising symmetry causes the correlator of an odd number of spin fields to vanish as well. The two-to-one map of λ -space to ϕ -space in the type 0B model highlights their nonlocal relation, a feature we have already seen several times.

This proposal passes checks analogous to the bosonic string – the tree level S-matrix, the torus partition function, and expectations of the ground ring operators on the sphere and on the disk, all agree between matrix and continuum approaches [22].

The \mathbb{Z}_2 symmetry that changes the sign of spin operators like $C_{i\omega}$, called *NSR parity*, also characterizes the boundary states, splitting them into \mathbb{Z}_2 even (*NS*) and odd (*R*) components. For instance, there are separate *NS* and *R* macroscopic loops. We may determine their functional form in the matrix model by repeating the calculation of section 7.1. The main differences will be that the calculation splits into these two boundary state sectors. The boundary state wavefunctions Ψ_{NS} and Ψ_R for both Dirichlet (ZZ) and Neumann (FZZT) branes appearing in (109) are given in [58, 59] (see also [22], sections 6 and 7). The matter partition function on the annulus [60] is essentially the same as equation (111), with the sum over n restricted to even integers in the NS sector and odd integers in the R sector. The

analysis then proceeds along the lines of section 7.1; one finds

$$\begin{aligned}\mathcal{Z}_{\text{NS}} &= -\frac{1}{2}\log[\mu_B^2 - \lambda^2(x_0)] + \frac{1}{2}\log(\mu/2) \\ \mathcal{Z}_{\text{R}} &= \frac{1}{2}\log\left[\frac{\mu_B - \lambda(x_0)}{\mu_B + \lambda(x_0)}\right].\end{aligned}\tag{123}$$

These results prove a conjecture [21, 61] for the form of the macroscopic loop operators in the matrix model for the type 0B fermionic string.

The super-Liouville boundary state wavefunctions [58, 59] are also the major ingredients of the disk one-point functions that yield the wavefunctions corresponding to the operators $V_{i\omega}$ and $C_{i\omega}$. For the tachyon, one finds essentially the same result (95), while for the RR scalar C , one finds

$$\langle C_{i\omega}(z, \bar{z}) \rangle_{\text{disk}} = \frac{1}{2\pi} \hat{\mu}^{-i\omega/2} (\Gamma(\frac{1}{2} + i\omega))^2 \cos(\pi s\omega). \tag{124}$$

The corresponding integral transforms to loop length wavefunctions again yield Bessel functions [21, 22, 61].

There are also a few discrete symmetries that match on both sides of the correspondence. One example is the $\lambda \rightarrow -\lambda$ parity symmetry of the matrix model, which appears as the \mathbb{Z}_2 NSR parity symmetry which sends $C \rightarrow -C$ in the continuum theory. The continuum theory also has a symmetry under $\mu \rightarrow -\mu$,³⁰ in the matrix model, this is the symmetry of the Hamiltonian $H = \frac{1}{2}(p^2 - \lambda^2)$ under $p \leftrightarrow \lambda$, combined with an interchange of particles and holes. A few other checks, as well as a second 2D fermionic string model – the type 0A string, whose matrix model formulation involves the dynamics of open strings in a system of D-particles and their antiparticles – can be found in [22].

8.2 Remarks on tachyon condensation

The structure of the bosonic and fermionic matrix models of 2D string theory is a remarkable illustration of the effective picture of tachyon condensation on systems of unstable D-branes [23]. In perturbative string theory, a D-brane is a heavy, semiclassical object much like a soliton. The analogy to solitons is in fact quite precise [62, 63, 64, 65]. Unstable D-branes are like solitons that do not carry a topological charge, and thus can decay to the vacuum (plus radiation). But being heavy, the initial stages of the decay are a collective process of instability of the ‘soliton’ field configuration. The quanta of this unstable mode are open string tachyons, and the initial stages of the decay are best described as the condensation of this tachyonic mode. An effective potential picture of this process is shown in figure 15a for an unstable brane in the bosonic string.

³⁰Combined with $\psi_{X,\phi} \rightarrow -\psi_{X,\phi}$ (but keeping $\bar{\psi}_{X,\phi}$ unchanged). This symmetry is the discrete \mathbb{Z}_2 *R-parity* symmetry of the worldsheet supersymmetry of the model.

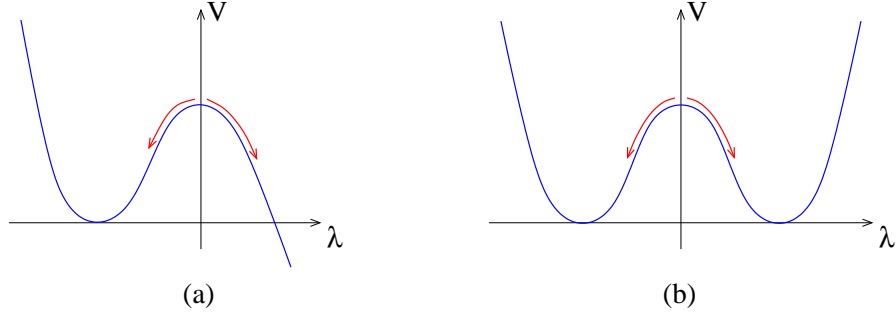


Figure 15: *The effective potential for the open string tachyon on an unstable D-brane in the (a) bosonic, and (b) fermionic strings.*

The heuristic picture of the effective potential identifies the local minimum to the left of the unstable point with the “closed string vacuum”, and the difference in energy between local maximum and local minimum is the energy of the initial unstable brane. An initial state of the tachyon field T localized at the unstable maximum of the effective potential is meant to describe the presence of the unstable brane, and condensation of T describes its decay. Condensation to $\langle T \rangle < 0$ represents decay toward the closed string vacuum. The abyss to the right of the local maximum is meant to represent the fact that condensing the open string tachyon to $\langle T \rangle > 0$ leads to singularities at finite time in perturbative calculations [66] with no known string interpretation; it is not understood whether there is any stable, nonsingular state to which the system evolves when the open string tachyon condenses in that direction.

Qualitatively, this picture is identical to that of the matrix potential of figure 8a. The only difference is that the closed string vacuum is itself described via the open-closed string equivalence as a degenerate gas of D-particles – in a sense a collection of unstable D-branes that have “already decayed”. The absence of eigenvalues to the right of the barrier means that there is no worldsheet interpretation for eigenvalues in this region, just like the region to the right of 15a.

A similar story applies to the fermionic string. The open string tachyon effective potential has two symmetric wells, as in figure 15b. On one hand, condensation in either direction of the open string tachyon on an unstable D-brane leads to its decay to the closed string vacuum; on the other hand, the matrix model for the type 0B string has just such a potential, with both wells filled by eigenvalue fermions, and a string interpretation of the physics on either side. The analogy also holds for the matrix model equivalent of the type 0A string.

9 Open problems

What remains to be understood? In this concluding section, let us list a few unresolved issues and directions for future work.

9.1 The open-closed string duality map

While there is a qualitative map (49) between λ -space and ϕ -space at the level of zero modes, a precise map between the matrix model and the full Liouville field theory remains to be worked out. This would require a complete translation between quantities in the matrix model and the Liouville (plus free scalar) field theory. One indication of a missing ingredient is that the asymptotic states of the matrix model have the leg poles of the continuum formalism stripped off, see equation (97). The poles incorporate the effects of discrete physical states in the continuum formulation [51]. While, as argued above, only center-of-mass string motion is physical at generic, continuous momenta, there is an additional discrete spectrum of physical states at special momenta [67, 68, 69, 43]; a simple example is the zero-momentum graviton vertex operator $V_{\text{grav}} = \partial X \bar{\partial} X$, which is manifestly physical since it is the action density for X . The continuum formalism knows how to incorporate gravitational effects, while these are currently put into the matrix model by hand; the matrix prescription for the S-matrix is to compute the LSZ-reduced density wave scattering amplitudes, and then multiply the result by a leg-pole factor $\frac{\Gamma(i\omega)}{\Gamma(-i\omega)}$ for each asymptotic state. It is this leg-pole factor which is responsible for perturbative gravitational effects [49, 50].

9.2 Gravitational effects

Perhaps a part of the explanation for this absence of gravitational and other discrete state effects in the matrix model is that, since the linear dilaton lifts the string “tachyon” mode to zero mass, it also raises the graviton to positive mass; its effects are subleading to the tachyon, and might be masked by or effectively absorbed into tachyon dynamics [70, 71].

Initially there was hope that the matrix model would teach us about nonperturbative gravity, and in particular lead to a solvable model of black hole dynamics. A second background solution to the string equations of motion (6)-(7) appears to be a black hole [72, 73]

$$\begin{aligned} ds^2 &= d\phi^2 \pm \tanh^2\left(\frac{1}{\sqrt{2(k-2)}}\phi\right) dx^2 \\ \Phi &= \Phi_0 + \log\left[\cosh\left(\frac{1}{\sqrt{2(k-2)}}\phi\right)\right], \end{aligned} \tag{125}$$

depending on whether we are interested in Euclidean or Lorentzian signature. The metric is written in Schwarzschild-like coordinates, where the horizon at $\phi = 0$ is infinitely redshifted relative to the asymptotic region $|\phi| \rightarrow \infty$. This sigma model

(5) on this background describes an exact conformal field theory, the $SL(2, \mathbb{R})/U(1)$ gauged WZW model (the signature is determined by the conjugacy class of the $U(1) \subset SL(2, \mathbb{R})$ being gauged). The level k of the $SL(2, \mathbb{R})$ current algebra symmetry of the WZW model is $k = 9/4$ for the bosonic string, and $k = 5/2$ for the fermionic string, in order that the slope $Q = \sqrt{\frac{2}{k-2}}$ of the asymptotically linear dilaton in (125) have the right value for 2D string theory. Note that the radius of curvature of the geometry is of order $1/\sqrt{k}$ in the vicinity of the horizon $\phi \sim 0$; therefore it is important to have an exact conformal field theory, since the corrections to the leading order equations of motion (7) are significant. Note also that the leading asymptotic perturbation $e^{2Q\phi}\partial X\bar{\partial}X$ of the metric away from flat spacetime, is the reflected version (the other on-shell value of Liouville momentum) of the special physical graviton operator $\partial X\bar{\partial}X$ discussed above. Thus the background can be thought of as the nonlinear completion of this linearized deformation. A shift in ϕ makes $e^{-2\Phi_0}$ the coupling in front of the asymptotic graviton in (125); as in higher dimensions, the coefficient of the leading asymptotic deformation of the metric away from flat spacetime is the mass of the black hole [74, 75], $\mu_{\text{bh}} = e^{-2\Phi_0}$.

A great deal is known about this CFT. There is a conformal bootstrap, analogous to that of Liouville theory [76, 77]. The analogue of the two degenerate operators $V_{-b/2}$, $V_{-1/2b}$ of Liouville theory are the degenerate operators Φ_j of $SL(2, \mathbb{R})$ current algebra, having spin $j = -\frac{3}{2}$ and $j = -\frac{k}{2}$.

A rather remarkable conjecture [78] claims that the Euclidean $SL(2, \mathbb{R})/U(1)$ gauged WZW model is equivalent as a quantum field theory to another model, the so-called Sine-Liouville theory, whose action is

$$SS_{\text{SL}} = \frac{1}{4\pi} \int \sqrt{g} \left[g^{ab} \partial_a \phi \partial_b \phi + g^{ab} \partial_a X \partial_b X + Q R^{(2)} \phi + \mu_{\text{sl}} \cos R[X_l - X_r] e^{\frac{1}{Q}\phi} \right]. \quad (126)$$

Here $Q^2 = \frac{2}{k-2}$; X is compactified on a circle of radius $R = 2/Q$; and $X_l - X_r$ is the axial component of X , so that the potential in (126) acts as a generating function for vortices in the worldsheet partition function. In [77], this equivalence is argued to hold at the level of the conformal bootstrap for correlation functions. The “resonant amplitudes”, which are those correlators dominated in the path integral by the asymptotic region $\phi \rightarrow -\infty$, involve only the operators $\Phi_{-\frac{r}{2}-\frac{s}{2}(k-2)-1}$, $r, s = 1, 2, \dots$. These correlators must in general be perturbatively dressed by *both* the asymptotic graviton $\mu_{\text{bh}} e^{2Q\phi} \partial X \bar{\partial} X$, which dresses r , and by the Sine-Liouville interaction $\mu_{\text{sl}} e^{\frac{1}{Q}\phi} \cos R(X_l - X_r)$, which dresses s . Self-consistency requires the coefficients of these two interactions to be related [77]; one finds [77]

$$\pi \mu_{\text{bh}} \frac{\Gamma(-Q^2/2)}{\Gamma(1+Q^2/2)} = \left(\pi \mu_{\text{sl}} Q^2/2 \right)^{Q^2}. \quad (127)$$

Again, as in Liouville theory there is a sense in which both dressing operators are present in the theory.

There is again a kind of strong/weak coupling duality, since the metric deformation is dominant at weak coupling ($\phi \rightarrow -\infty$) for $Q \ll 1$, while the Sine-Liouville coupling is dominant for $Q \gg 1$. Since $Q = 2$ for the 2D string, one has the sense that the Sine-Liouville description is somewhat more appropriate. In higher dimensions, when the curvature of a black hole reaches string scale, it undergoes a phase transition to a gas of strings [79] (the transition point is known as the *correspondence point*). The apparent dominance of the Sine-Liouville coupling may be an indication that the “black hole” of 2D string theory is actually on this other side of the correspondence point, where it is better thought of as a gas or condensate of strings.

The equivalence with Sine-Liouville leads to a natural candidate [80] for a matrix model equivalent to the Euclidean “black hole” – simply turn off the Liouville potential and turn on a condensate of vortices in the compactified Euclidean theory, *c.f.* section 4.1. The matrix description of the background thus has a closer affinity to the tachyon condensate of (126) than it has to the Euclidean black hole of (125).³¹

Yet another reason to suspect the absence of objects that could truly be characterized as black holes in 2D string theory, is the absence of nonsinglet states in the matrix model. As mentioned in the introduction, the appearance of black holes in the density of states in higher dimensional versions of the gauge/gravity equivalence is associated to a deconfinement transition. The thermodynamics one is led to [74, 75] on the basis of the classical gravity solution (125) yields a density of states $\rho = \exp[\sqrt{2}\pi E]$. Such a density of states will not come from the quantum mechanics of the degenerate Fermi gas of the singlet sector of the matrix model, but might conceivably come from the liberation of nonsinglet degrees of freedom of the matrix. However, this is absent from the matrix model – the $U(N)$ degrees of freedom are gauged away.

Indeed, a calculation [83] of nonperturbative high energy scattering in the matrix model – a process that in higher dimensions would certainly lead to the formation of black holes as long-lived intermediate states – reveals none of the features that would be predicted on the basis of the appearance of black holes being formed during the scattering process.

In short, low energy gravitational effects are put into the matrix model by hand, via the leg-pole factors. High energy gravitational effects such as black hole formation seem to be absent altogether. Does the matrix model incorporate any form of 2d gravity? If so, how? If not, why not?

9.3 Short-distance physics

Even though it would appear that black hole physics is absent from the matrix model, intriguing remnants of Planck scale (or more precisely, ultra-short distance)

³¹After T-duality ($X_r \rightarrow -X_r$) and Wick rotation to Lorentz signature, the background (126) describes a moving tachyon wall $V_{\text{backgd}} = \mu_{\text{sl}} e^{\phi/Q} \cosh(RX)$. Such time-dependent tachyon condensates have been considered in [81, 82].

physics seem to be present. Namely, the spacing of eigenvalues in the matrix model is of order the D-particle Compton wavelength $L_c \sim e^\Phi \ell_s$.

The fact that loop length scales as $\ell \sim e^{b\phi}$, together with the integral transform (49), suggests that the eigenvalue coordinate scales as $\lambda \sim -e^{-b\phi}$ (in the sense of KPZ scaling). From equations (51) and (65) one determines $\langle \hat{\rho} \rangle \sim |\lambda|$ as $\lambda \rightarrow -\infty$. The eigenvalue spacing is $\delta\lambda \sim 1/\langle \hat{\rho} \rangle$, and thus $\delta\lambda/\lambda \sim \lambda^{-2}$. In terms of the Liouville coordinate, this spacing is $\delta\phi \sim e^{2\phi} = e^\Phi$, which is L_c ! This result generalizes to the discrete series of $c < 1$ conformal field theories coupled to Liouville gravity, which are thought of as string theory in $D < 2$. Here we have $b = \sqrt{q/p}$, with $p, q \in \mathbb{Z}$ and $q < p$. The pair (p, q) characterize the matter conformal field theory, with $c_{\text{matter}} = 1 - 6\frac{(p-q)^2}{pq}$. These models have a realization as an integral over two random matrices [84, 85, 86] with the eigenvalue density scaling as $\rho(\lambda) \sim \lambda^{p/q}$. Tracing through the KPZ scaling, one finds $\delta\lambda/\lambda \sim \lambda^{-(1+1/b^2)}$, and once again the eigenvalue spacing is $\delta\phi \sim e^{Q\phi} = e^\Phi$. An appealing interpretation of this result is that spacetime has a graininess or discrete structure at the short distance scale L_c . It would be interesting to find some ‘experimental’ manifestation of this spacetime graininess.

9.4 Open string tachyons

In higher-dimensional spacetime, the canonical picture of the decay of unstable D-branes has the initial stages of the decay well-described by open string tachyon condensation; at late times the brane has decayed, open strings are absent, and the energy is carried off by a pulse of closed string radiation.

The qualitative picture is rather different in 2D string theory. Here the branes don’t really decay; the open string tachyon merely describes their motion in spacetime, and there is an *equivalence* between two characterizations of the dynamics in terms of open or of closed strings.

The worldsheet formulation has elements of both open and closed string descriptions of D-brane decay. Closed string worldsheets represent the collective dynamics of the Fermi sea of “decayed” eigenvalues; eigenvalues extracted from the sea are represented as D-branes with explicit open string degrees of freedom. Thus the continuum description is naively overcomplete. For instance, one can compute the “radiation” of closed strings from the “decaying” D-brane representing an eigenvalue rolling off the potential barrier [18, 20]. One finds a closed string state

$$|\psi\rangle \sim \exp\left[i \int d\omega v_p \alpha_p^\dagger\right] |\text{vac}\rangle \quad (128)$$

with the coefficient v_p given to leading order by the disk expectation value of the tachyon vertex operator $V_{i\omega}$ with the boundary conditions (107). Roughly, the closed string tachyon bosonizes the eigenvalue fermion.

Of course, the eigenvalue doesn’t decay, but stays in its wavepacket as it propagates to infinity. An eigenvalue fermion maintains its identity as it rolls to infinity;

we are not forced to bosonize it. There appears to be some redundancy in the worldsheet description, unless different descriptions are valid in non-overlapping regimes (as is the case in other open/closed string equivalences); but then it remains to be seen what effects force us to describe the dynamics as that of D-branes or that of closed strings, and in what regimes those effects are important. A possible clue is the form of the D-brane boundary state, which fixes the boundary at $\phi = \infty$ throughout the motion, and instead describes the dynamics as occurring in the field space of the open string tachyon. On the other hand, we know that $\lambda \rightarrow -\infty$ corresponds to $\phi \rightarrow -\infty$, and therefore at late times an appropriate boundary state should have significant support in this weak coupling region. This suggests that the perturbative boundary state description of the rolling eigenvalue breaks down at finite time.³² It was pointed out in [87] that the boundary state represents a source for closed strings that grows exponentially in time, so that one would expect the perturbative formalism to break down at a time of order $x \sim \log \mu$ (note that this is roughly the WKB time of flight from the top of the potential to the edge of the Fermi sea). Once again we run into the issues surrounding the quantization of the D-brane motion mentioned at the end of section 7.1.

A similar issue is the absence so far of a completely convincing worldsheet description of holes in the Fermi sea of eigenvalues (for a proposal based on analytic continuation of the boundary states, see [22, 88]). Holes lie within the Fermi sea instead of being separated from it, and so all the questions as to when and whether there is an open string description apply here as well. The worldsheet description of holes is an important missing entry in our translation table.

It is interesting that the open/closed string equivalence in this system is built out of objects that don't carry conserved charge, as opposed to standard examples like D3 branes providing the gauge theory dual to $AdS_5 \times S^5$, which are charged sources for antisymmetric tensors $C_{(r)}$. It raises the question of whether there are other situations in string theory where there is an open-closed string equivalence in terms of uncharged objects. A good part of the program to understand open string tachyon condensation is driven by this question. Is the late-time dynamics of the open string tachyon condensate on unstable branes (sometimes called *tachyon matter*) an alternate description of (at least a self-contained subsector of) closed string dynamics? We have one system where the answer is yes, and it would be interesting to know if there are others, and if so whether such an equivalence holds generically (*c.f.* [18] for a discussion in the present context).

9.5 Closed string tachyons

Although the linear dilaton lifts the mass shell of the “tachyon” to zero in 2D spacetime, the spacelike tachyon condensate of 2D string theory may still contain clues to the properties of closed string tachyons in string theory. While much of the physics of open string tachyon condensation is relatively well understood by now,

³²I thank Joanna Karczmarek for discussions of this issue.

closed string tachyon condensation is still rather mysterious. The only controlled examples which have been studied involve closed string tachyons on localized defects [89, 90, 91, 92] (for reviews, see [24, 93]). In these cases, the localized defect decays to flat spacetime with a pulse of radiation, much like the decay of D-branes via open string tachyon condensation. The condensation of delocalized tachyons is less well understood. The resulting backgrounds will have a cosmological character since the spacetime geometry will react to the stress-energy density of the evolving tachyon field.

Examples of this sort are just beginning to be studied in 2D string theory. In a sense, the closed string tachyon condensate is really only a stationary rather than a static background of the continuum theory. From the open string point of view, the custodian of this 2D cosmos must sit with a bucket of eigenvalues and keep throwing them in at a constant rate in order to preserve the Fermi sea. If this entity tires of its task, the Fermi sea drains away; the corresponding closed string background is then a time-dependent tachyon field. Properties of such backgrounds have been investigated in [81, 82, 94, 95, 96, 97], and might serve as a paradigm for the general problem of closed string tachyon condensates.

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